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1. F. A. Osborn: Change of Index of Refraction of Water with Change of Temprature.
   (Represented from *Physical Review,* March 1913)

2 P. Field: On Constrained Motion.
   (Reprinted from *Zeitschr. f. Math. und Physik,* March 1913)

3. N. H. Williams: The Stability of Residual Magnetism.
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4. N. H. Williams: Anomalous Temperature Effects upon Magnetized Steel.
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5. C. W. Green: Polarization of the Aluminium Rectifier.
   (Reprinted from *Physical Review,* April 1914)

6. G. A. Lindsay: A Study of the Longitudinal Vibration of Wires.
   (Reprinted from *Physical Review,* June 1914)

7. K. E. Guthe: Roger Bacon as a Scientist.
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CHANGE OF INDEX OF REFRACTION OF WATER WITH CHANGE OF TEMPERATURE.

BY FREDERICK A. OSBORN.

INTRODUCTION.

THE change of the index of refraction of water with change of temperature has been studied by a number of investigators, among whom the following may be mentioned:

Jamin[1] made observations on the change of the index of refraction of water between 0° and 30°. He used an interference method, but apparently did not publish any values. He contented himself with a general statement of the change and gave the empirical formula

$$K_t = K_0 - (0.000012573t) - (0.000001929t^2)$$

as expressing his results.

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Gladstone and Dale,[2] soon after the work of Jamin, studied the same problem, making use of a hollow prism and the method of minimum deviation. They worked between the temperatures of 0° and 40° and remark "our determinations were performed repeatedly and most carefully on water near the freezing point." They call attention to the fact that the change of index between 0° and 5° is much less than that between 5° and 10°, finding from their work that the change is .0001 and .0002 respectively. Their method enabled them to express the change of index to the fourth decimal place.

Ruhlman,[3] by a method similar to that of Gladstone and Dale, obtained the index of refraction of water between 0° and 100° for the lithium, sodium and thallium lines. His values are given to five decimal places.

Lorenz[4] used an interference method and worked between 0° and 34°. $d\mu/dt$ was found from the shift of the fringes with change of the temperature and an accuracy of 3 units in the seventh decimal place was claimed. No experimental data were given but his results are expressed by the equation

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[1] Comptes Rendus, 43, p. 1191, 1856.

[2] Phil. Trans., p. 887, 1885.

[3] Pogg. Ann., 132, p. 1177, 1867.

[4] Wied. Ann., 11, p. 70, 1880.

http://www.archive.org/details/contributions02michuoft

$$\frac{d\mu}{dt} = -10^{-6}(0.076 - 5.606t + 0.06403t^2) \text{ for the Na line,}$$

$$= -10^{-6}(0.952 - 5.586t + 0.06402t^2) \text{ for the Li line.}$$

Dufet[1] used two different methods. In the prism method, the prisms filled with water at a given temperature, were adjusted to minimum deviation. The water was then exchanged for water at a higher temperature, and the displacement of the lines was observed as the temperature fell. From this shift of the lines, and the other measurable quantities, he computed $d\mu/dt$. In the interference method he used Talbot's fringes, and observed the passage of about 180 fringes between 0° and 50°. In part of this work a quartz plate was immersed in the water, and for part a crown glass plate was substituted. The values for $d\mu/dt$ obtained by his different methods sometimes differ from each other by as much as 45×10^{-7}. Dufet expresses his results by the equation

$$\frac{d\mu}{dt} = -10^{-7}(125.46 + 41.285t - 0.01304t^2 - 0.0046t^3).$$

Values for $d\mu/dt$ obtained from this equation occasionally differ from his mean experimental values by as much as 7×10^{-7}.

Kettler[2] repeated Ruhlman's work for temperatures above 20° using a total reflection refractometer.

Walter,[3] using the minimum deviation method, has done some careful work for the range 0° to 30°. His results are given to the fifth decimal place.

Conray,[4] by the same method, but with a spectrometer graduated to 10″, has made a study of the change of index between 0° and 10°. He claims an accuracy in the sixth decimal place, although greater differences between successive determinations for the same temperature range are found.

Bender,[5] with the Pulfrich refractometer, has studied the change of the index of refraction of water between 0° and 40° using the H_α, H_β, H_γ lines. His values are expressed to five decimal places.

More recently Gifford,[6] using a new prism method, has determined the index at 15° for a large number of wave-lengths. The values obtained are considered correct to within two units in the fifth decimal place. A still later determination by Baxter, Burgess, and Daudt[1] gives the value of the index for Na light at the temperatures 20°, 25°, and 30°. These investigators used the prism method also and consider their results correct to three units in the fifth decimal place.

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[1] J. de Phy., p. 401, 1885.

[2] Wied. Ann., 33, p. 353, 506, 1888.

[3] Wied. Ann., 46, p. 422, 1892.

[4] Proc. Royal Soc., 58, p. 228–234, 1895.

[5] Ann. d. Phy., p. 343, 1899.

[6] Roy. Soc. Proc., 70, p. 329, 1902; 78, p. 406, 1906–7.

While the change of the index of refraction of water with change of temperature has been studied by these and many others, the methods used and the results obtained are such as to make it seem desirable to repeat the work.

It was thought that the Abbe-Pulfrich interferometer could be made to give more accurate results than those thus far obtained and the present paper has for its object the application of the interferometer to the determination of the change of the index of refraction of water with change of temperature.

METHOD.

The method used in this piece of work is that first used by Reimerdes[2] in his work on the change of the index of refraction of quartz. The change in the index of the liquid is determined by the shift of interference fringes in a fused quartz system.

The arrangement of the system is shown in Fig. 1. A bed plate *A*, of fused quartz, 38.4 mm. in diameter and 10 mm. thick, with its upper face plane polished, is placed on a horizontal surface. Upon the quartz surface is placed a fused quartz ring *Q* of 31.8 mm. in external diameter, 24.4 mm. internal diameter, and approximately 9.9 mm. high. The cover plate *B*, of fused quartz, is approximately 36 mm. in diameter and 8 mm. thick. Its surfaces are plane polished and inclined to each other at an angle of about 20″, so as to deflect from the field the light reflected from its upper surface.

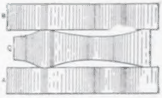

Fig. 1.

The quartz ring was cut from a plate of quartz whose surfaces were plane polished and slightly inclined to each other. The ends of this ring were ground away in such a manner as to leave on each end three equidistant feet, whose surfaces are portions of the original plane polished surfaces of the plate from which the ring was cut. The inclination of the faces of this plate to one another was such that when the ring is placed, as in the figure, between the bed and cover plates the opposite reflecting surfaces, *i. e.*, the upper surfaces of the bed plate and the lower surface of the cover plate, are properly inclined to produce suitable interference fringes.

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[1] Am. Chem. Soc. Jour., 33, p. 893, 1911.

[2] Inaugural Dissertation, Jena, 1896.

The cover plate B has on its lower surface at the center a small circle which serves in the field of view as the reference point from which the measurements are made.

When the apparatus is in adjustment and the field illuminated by monochromatic light, the field is seen to be crossed by parallel interference fringes. Each dark band represents a definite thickness of the air between A and B. If λ be the wave-length of the light used, then the difference in the thickness of the air layer for any two adjacent dark bands is $\lambda/2$. If the space between A and B be filled with a liquid, the effect is to increase the optical thickness between A and B from d to μd and the distance between any two fringes, the band width, is less than the distance between these same fringes in air.

When the fused quartz system and the liquid are heated, the fringes move across the field toward the thick edge of the wedge. The index of refraction of the water decreases with rise of temperature, hence the wave-length in the water becomes longer. There will be fewer wave-lengths between any two opposite points of the wedge. But a given fringe is due to the retardation of a definite number of wave-lengths of the light reflected from the upper surface of the bed plate over that reflected from the lower surface of the cover plate, so if the wave-lengths become longer, a thicker part of the wedge is needed to give the same number of wave-lengths retardation. Therefore, the bands move toward the thick edge of the wedge and the number passing the reference circle indicates the decrease in the number of waves between the surfaces at that point due to the lengthening of λ.

The expansion of the quartz ring increases the distance between the plates and decreases the number of fringes that would otherwise cross the reference circle toward the thick edge of the wedge, or increases the number of wave-lengths under the reference circle, which would cause a shift of the fringes toward the thin edge of the wedge. The resultant shift of fringes is the difference of these two effects and, owing to the small expansion of the fused quartz, is toward the thick edge of the wedge.

The change in the index of refraction for a given temperature change is given by the formula

$$\frac{d\mu}{dt} = -\frac{N\frac{\lambda}{2}}{L_1(t_2 - t_1)} - \alpha\mu_1.$$

That this is so appears from the following considerations.

Let n be the number of fringes shifted for the temperature interval $(t_2 - t_1)$.

L_1 and L_2 be the thicknesses of the liquid (length of ring) at t_1 and t_2.

t_1 and t_2 be the lower and higher temperatures.

μ_1 and μ_2 be the indices of refraction of the liquid at t_1 and t_2.

λ be wave-length in air of the light used.

λ_1 and λ_2 be corresponding wave-lengths in the liquid at t_1 and t_2.

α be the mean coefficient of expansion of the fused quartz ring for the given temperature interval $(t_2 - t_1)$.

$$\mu_1 = \frac{\lambda}{\lambda_1} \text{ and } \mu_2 = \frac{\lambda}{\lambda_2}, \quad \lambda_1 = \frac{\lambda}{\mu_1} \text{ and } \lambda_2 = \frac{\lambda}{\mu_2},$$

$$n = \frac{2L_1}{\lambda_1} - \frac{2L_2}{\lambda_2} = \frac{2L_1}{\frac{\lambda}{\mu_1}} - \frac{2L_2}{\frac{\lambda}{\mu_2}} = \frac{2L_1}{\frac{\lambda}{\mu_1}} - \frac{2L_1[1 + \alpha(t_2 - t_1)]}{\frac{\lambda}{\mu_2}},$$

$$= 2L_1\frac{(\mu_1 - \mu_2)}{\lambda} - \frac{2L_1}{\lambda}\mu_2\alpha(t_2 - t_1),$$

$$\mu_2 - \mu_1 = -\frac{n\frac{\lambda}{2}}{L_1} - \mu_2\alpha(t_2 - t_1),$$

$$\frac{d\mu}{dt} = \frac{\mu_2 - \mu_1}{t_2 - t_1} = -\frac{n\lambda}{2L_1(t_2 - t_1)} - \mu_2\alpha,$$

$$= -\frac{n\lambda}{2L_1(t_2 - t_1)} - \mu_1\alpha,$$

as the difference between μ_1 and μ_2 is not sufficiently large to affect the value of the corrections.

Apparatus.

The instrument used was an Abbe-Pulfrich interferometer such as described in detail by Pulfrich[1] and also by Randall.[2] The source of the light was a small mercury arc lamp, giving a very brilliant light in which the line in the green was especially prominent.

The quartz system was set up in a glass cell, 7 cm. in diameter and 4 cm. deep, which was placed in the double-walled galvanized iron tank, *T*. The cover, *G*, closed the tank and made it water-tight by means of mercury between the double walls. The small opening in *G* at the top, 3 cm. in diameter, was closed by a glass plate. A Beckmann thermometer, *B*, and a stirrer, *S*, were introduced into the cell through the cover as shown in Fig. 2. The tank with its optical system was rigidly supported by the arm *L*, which was clamped to the stone pier on which the interferometer rested.

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[1] Zeitschr. fur Instrumentenkunde, 1898, p. 261.

[2] Phys. Rev., 20, 1905.

The heating was obtained by means of two coils of No. 28 iron wire wound on the insulated vessel *V*, which was 18 cm. in diameter and 25 cm. deep. The coils, shown by the small circles, were wound separately, the turns about 5 mm. apart, so that the coils could be used alone, in series, or in parallel. The coils were covered with a layer of asbestos and then with wool felt. This vessel was made to fit closely into another iron vessel which was set inside a larger tank 35 cm. in diameter and 40 cm. high. The space between the two could be filled with water or with ice.

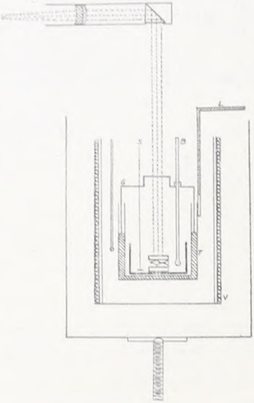

Fig. 2.

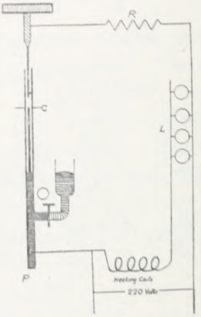

Fig. 3.

The thermostat used was a slight modification of a form which has already been described by the writer.[1] Around the inside of *V* was a liquid thermometer of 180 c.c. volume. This thermometer consisted of a ring of glass tubing, 17 cm. in diameter, with six thin-walled vertical glass tubes, 20 cm. long, fused into it and hanging downward, all filled with toluol, except one which was filled with mercury and connected with the regulator at *P*, Fig. 3.

Fig. 3 shows the regulator and the electrical connections. The heating current was regulated by a lamp bank, *L*, not more than six 16 c.p. 220-volt lamps being required for the highest temperature. The thermostat was adjusted for a given temperature, by making the resistance in the rheostat, *R*, of such value that the current in the heating coils was insufficient to raise the temperature of the bath; the latter, cooling slightly, caused the mercury in the capillary, *C*, to fall, thus cutting out the rheostat circuit, and allowing more current to go through the heating coils. When the temperature of the bath rose, the mercury also rose in the capillary, cutting in the rheostat.

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[1] Jour. Phys. Chem., 9, 1905, p. 207.

Since the large volume of the liquid thermometer made it more sensitive than the thermometer in *V*, the bath was easily held at a temperature constant to within 2 or 3 thousandths of a degree for half a day or more. The consequent variation of the temperature in the cell was not more than .002° in the same time.

Stirrers in *T* and *V* kept the water thoroughly agitated.

Adjustment of Apparatus.

The preliminary adjustments of the instrument and the optical system for work with the fringes has been described by Pulfrich.[1]

When the system was in good adjustment, the liquid whose index of refraction was to be studied was poured into the glass cell, until, when the cover plate *B* was put on the quartz ring, the liquid came within a mm. of the top of *B*. The cover *G* was then put on and the entire bath raised by means of a screw until the water around the tank *T* came within a cm. of the glass top on *G*. The temperature of the bath was then raised to the desired point and the thermostat set in operation by closing the stop cock *O* and adjusting the platinum point in the capillary tube.

The temperature in the cell would gradually rise for two or three hours, after which the Beckmann thermometer showed no change of temperature greater than .002°. When the temperature had been constant for two hours longer, the position of the bands was read and also the temperature. One hour later both were read again. If no change in either was found, the temperature of the liquid was assumed to have become constant.

The temperature was then slowly raised 5° and the displacement of the fringes was counted. This displacement was, after the first ten minutes, very uniform, until the temperature of the cell was within 0.5° of that of the surrounding water bath when the displacement became slow. The entire time for this displacement of the fringes was usually about two hours. But, as in getting the initial position of the bands, the final reading was not taken until the thermometer had been constant to within .002° for three hours. The number of whole bands shifted being known by actual count, the total shift was then given by the formula

$$N = M + d_2 - d_1,$$

where M = number of whole bands counted, d_2 and d_1 the fractions of a band at the higher and lower temperatures respectively. These fractions were found from the readings of the micrometer with which the interferometer was provided.

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[1] Zeitsch. fur Instrumentenkunde, 13, Oct., Nov., Dec., 1893.

THERMOMETERS.

Two thermometers were used: For the water bath, thermometer No. 2,107, range − 10° to 50° by 1/10°, recently calibrated at the National Bureau of Standards; for the liquid in the cell a Beckmann thermometer, compared with No. 2,107, and, for the 5° interval, compared with one made by Haak, reading to .01° which had been calibrated at the Reichsanstalt.

The mean temperature of the water in the cell was obtained in each case by translating the readings of the Beckmann thermometer into degrees centigrade.

SPHEROMETER.

A Geneva Society spherometer was used to measure the quartz ring, and the readings could be made to .001 mm. A mean of thirty readings gave for the length of the ring 9.901 mm.

DEGREE OF ACCURACY.

From the formula

$$\frac{d\mu}{dt} = -\frac{n\lambda}{2L_1(t_2 - t_1)} - \alpha\mu_1$$

it is evident that the value of $d\mu/dt$ depends upon the three measurable quantities, n, L, and t.

In finding n from the formula

$$n = M + d_2 - d_1,$$

the d's can be found with an error affecting n by ± .02 of a fringe. The resulting error in $d\mu/dt$ is 10×10^{-8}. The difference in temperature could be determined with an error of ± .002. The error in $d\mu/dt$ from this source is 4×10^{-8}. The length of L could be found to ± .001 mm., resulting in an error in $d\mu/dt$ of 1×10^{-8}.

The total error in the final result due to these errors is

$$E = 10^{-8}\sqrt{10^2 + 4^2 + 1^2} = 10.8 \times 10^{-8},$$

or 1 unit in the seventh decimal place.

Scheel has found the length of a fused quartz ring similar to the one used here to be well represented by the expression

$$l_t = l_0(1 + 0.322 \times 10^{-6}t + 0.00147 \times 10^{-6}t^2).$$

The mean value of α computed from the above formula for each temperature range was used; and μ_1, as found by the prism method, using a Zeiss spectrometer, is 1.3342. μ_1 need not be known with the greatest accuracy as an error of .001 is negligible as affecting the factor $\alpha\mu_1$.

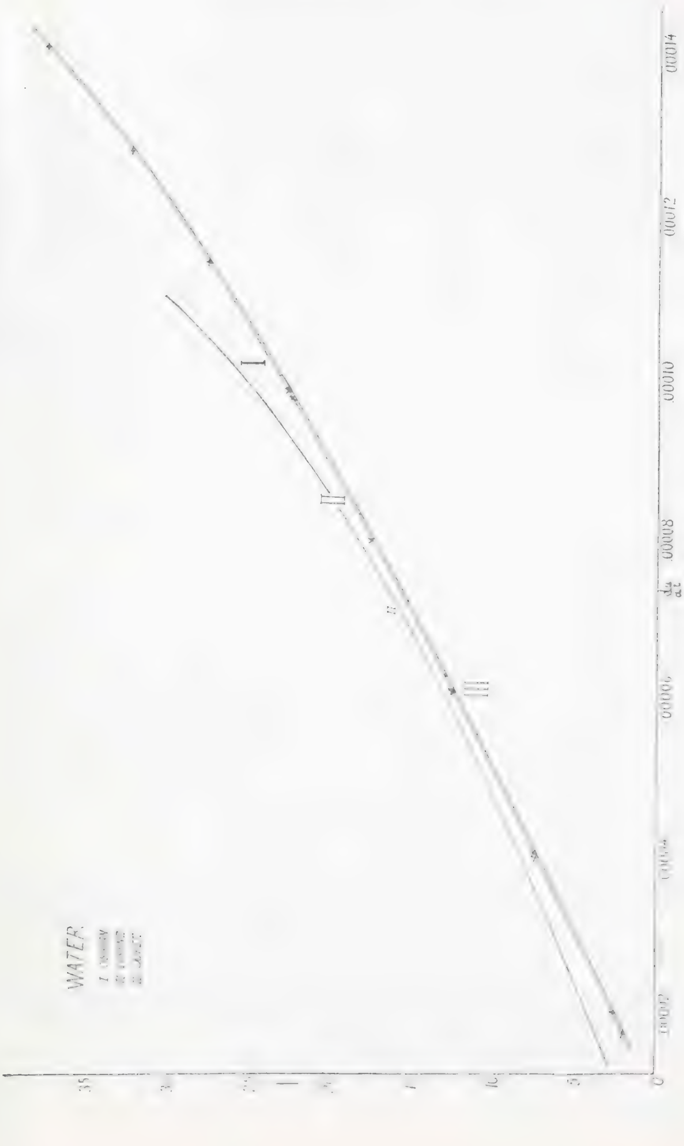
WATER.
I
II
III
0
5
10
35
.0002
.0004
.0006
.0008
.0010
.0012
.0014
$\frac{d\mu}{dt}$

Fig. 4.

EXAMPLE OF A COMPLETE DETERMINATION.

Length of fused quartz ring.................................9.901 mm.
Thickness of water layer....................................9.901 mm.
Conductivity of water used (free from dissolved air)..........1.7×10^{-6}

Time.	Temperature of Bath.	Temperature of Water in the Cell, Beckmann Thermometer.	Temperature of Emergent Column of Beckmann.	Shift of Fringes.	d's from Micrometer Readings.
9:30A.M.	25.00° C.	1.150			12 noon
1:00		.170	23.5	0	
2:00		.190			$d_1=.190$
1:00		.190			1:30 P.M.
1:30		.190			$d_2=.186$
					Mean .188
1:45	25.5			1	
50		1.900		1	
54				1	
58				1	
2:02				1	
06				1	
10				1	
14				1	5:00 P.M.
18				1	
22				1	
26				1	$d_2=-.047$
29				1	
33				1	
37				1	
40				1	6:30 P.M.
44				1	
48				1	$d_2=-.043$
54				1	
59				1	
3:16	29.98			1	Mean $-.045$
4:00		6.120	25.5	1	
5:00		.130		21	
30		.130			
6:30		.130			

$$N = M + d_2 - d_1 = 21 + (-0.045) - 0.188 = 20.767,$$

$$t_2 - t_1 = 6.130 - 1.190 = 4.940°,$$

$t_2 - t_1$ corrected for emergent stem and mercury out of bulb $= 4.999°$,

$$\frac{d\mu}{dt} = -\frac{20.767 \times .0005460}{2 \times 9.901 \times 4.999} - .00000048 = -.0001150$$

at 27.49°.

Table I.

Water.

Conductivity of water: 1.7 × [illegible] [illegible].
$L = 9.001$ mm.
$\lambda = .0005460$ mm.

Date.	Series.	Temp. Range.	Temp. Diff. $t_2 - t_1$	Total Shift of Fringe.	$\frac{d\mu}{dt}$	At
March 23	1	20–25	5.014	17.95	.0000992	22.58°
25	1	20–25	4.992	17.90	983	22.33
26	2	25–30	4.999	20.767	1150	27.49
27	2	30–25	4.981	20.684	1150	27.51
28	2	30–35	4.952	22.978	1284	32.25
28	2	35–30	4.860	22.540	1284	32.23
April 2	2	35–40	5.015	25.541	1409	37.50
3	2	40–35	5.013	25.528	1409	37.50
5	3	15–20	4.934	14.393	809	17.37
6	3	20–15	4.860	14.156	808	17.38
6	3	15–10	4.965	11.170	625	12.49
8	3	10–5	4.990	7.617	424	7.49
8	3	5–0	4.714	3.805	227	2.61
11	4	10–15	5.009	11.284	624	12.50
11	4	15–20	5.007	14.607	809	17.37
12	4	20–25	4.961	17.786	993	22.53
12	4	25–30	4.893	20.331	1150	27.46
13	4	30–35	5.052	23.429	1284	32.27
13	4	35–40	4.910	25.017	1410	37.48
14	4	5–10	4.730	7.271	420	7.34
14	4	5–0	4.495	3.632	227	2.59

Table I. gives the complete set of experimental data. Each series represents a new set-up of the apparatus and a new supply of water taken from a common source.

Fig. 4, Curve I., expresses the results graphically, and they may also be given by the equation

$$\frac{d\mu}{dt} = -10^{-7} \times (118.73 + 41.4184t - 0.02376t^2 - .0043757t^3).$$

This equation expresses correctly the experimental results to approximately one unit in the seventh decimal place.

Conclusion.

The results contained in Table I. show that the method here employed gives more concordant results than any method previously used and that in the determination of $d\mu/dt$ an accuracy of one unit in the seventh decimal place has very approximately been attained. This result has been achieved principally through the accuracy with which the shift of the

TABLE II.

Comparison of Results.[1]

Temp.	Lorenz.	Dufet.	Osborn.	Temp.	Lorenz.	Dufet.	Osborn.
38°		.0001423	.0001418	19°	.0000833	.0000874	.0000867
37		1402	1398	18	801	838	831
36		1380	1375	17	767	801	794
35		1357	1352	16	732	764	757
34		1333	1327	15	696	726	720
33		1308	1301	14	659	688	682
32		1282	1276	13	620	650	644
31		1256	1249	12	580	611	605
30	.0001105	1228	1221	11	538	572	565
29	1087	1200	1194	10	496	532	526
28	1067	1170	1163	9	452	493	486
27	1045	1140	1134	8	407	453	447
26	1024	1109	1103	7	360	412	406
25	1001	1078	1072	6	313	372	365
24	976	1045	1039	5	264	331	325
23	950	1012	1006	4	213	290	284
22	923	978	972	3	162	249	243
21	894	944	938	2	109	208	201
20	864	909	903	1	55	167	160

interference bands could be measured by the Abbe-Pulfrich interferometer and also by the extremely exact control and measurement of the temperature.

In comparing the results here obtained with those of other observers, it will be noticed from the results tabulated in Table II. and also from the Curves I. and III. that a constant difference exists between the results of Dufet and those of the writer. This difference is six or seven units in the seventh decimal place. The experiments of Lorenz furnish data for estimating the variation of $d\mu/dt$ with respect to λ. The change in $d\mu/dt$ to be expected for a change from the Na line, λ 5,893, used by Dufet to the Hg line, λ 5,460, used by the writer, is, according to Lorenz, 6.8 units in the seventh decimal place in the temperature region 10°–30°. The results of the two investigations appear therefore to be in complete accord. This is somewhat remarkable when one considers that each of Dufet's final results is the mean of four different determinations which often differ from one another by several units in the sixth decimal place.

The empirical equation here given expresses the experimental results to one unit in the seventh decimal place. It is given in preference to adopting the Dufet equation with a suitable alteration of its constant term as it agrees more closely with the experimental data.

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[1] Computed from the empirical equations.

Gifford's data on the change of the temperature refractive coefficients with wave-length are not sufficiently explicit to warrant an estimate of the difference which should exist between the values here presented and those of Dufet.

The changes in the index for the temperature interval 20° to 25° and 20° to 30° as computed by the formulæ of Dufet and the writer, the later values being corrected for wave-length, *i. e.*, to Na 5,893, agree with the recently measured changes according to Baxter, Burgess, Doudt, as the following table shows.

	B., B. & D. Observed.	Dufet Comp.	Osborn Comp.
20°–25°	.00051	.0004973	.0004975
20°–30°	.00109	.0010745	.0010748

The differences all lie within the estimated errors of observation.

To Professor Reed, under whose direction this work was done, the writer wishes to express his thanks.

Physical Laboratory,
University of Michigan.

Sonderabdruck aus Zeitschrift f. Mathematik u. Physik. 62 Band. 1913. Heft 3.
Druck und Verlag von B. G. Teubner in Leipzig.

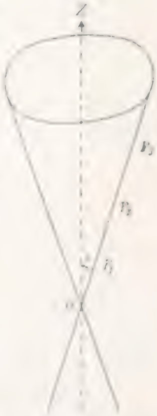
On Constrained Motion.

By Peter Field, Ann Arbor, Mich., U.S.A.

Introduction.

M. M. A. de Saint-Germain and L. Lecornu have given in Vol. CXIV of the Comptes Rendus the following example to illustrate that a motion which may be possible geometrically may be impossible dynamically. Three equal beads are attached to a weightless rod, one at each end and the third at the mid point. The beads and rod are constrained to remain on the surface of a smooth right circular cone. The problem is to determine the motion for any given initial conditions when no external forces act on the particles.

Fig. 1.



Call r, α, φ the polar coordinates of P_2 (Fig. 1), $P_1P_3 = 2a$, and indicate derivatives with respect to t by dots.[1] The kinetic energy of the three particles and also the angular momentum about OZ are constant, so we have

$$3\dot{r}^2 + (3r^2 + 2a^2)\sin^2\alpha\,\dot{\varphi}^2 = A,$$
$$(3r^2 + 2a^2)\dot{\varphi} = B.$$

From the principle of the motion of the centroid

$$3\dot{r}^2 + 3r^2\sin^2\alpha\,\dot{\varphi}^2 = C, \qquad 3r^2\dot{\varphi} = D.$$

Resolving along the radius vector

$$\ddot{r} - r\dot{\varphi}^2\sin^2\alpha = 0.$$

These equations can only be satisfied by $\dot{\varphi} = 0$, $\dot{r} =$ a constant if $r \neq 0$. If then the centroid is not at the vertex of the cone, the rod moves uniformly along the same generator no matter what the initial conditions even though there is no geometrical reason why it should not move from one generator to another. As stated by Routh loc. cit., if the motion is not initially along a given generator an impulsive couple acts with the result that the rod moves along a generator.

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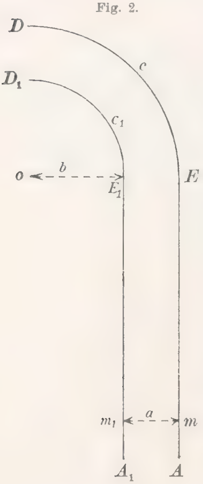
[1] Routh, Advanced Rigid Dynamics, sixth edition, p. 206.

This result can also be explained as follows. The projections of the velocities of P_1, P_2, P_3 along the tangents to the circular sections of the cone at P_1, P_2, P_3 are in the same ratio as the distances of P_1, P_2, P_3 from the vertex of the cone. As the reaction of the cone is not normal to its axis, it follows that the ratios of these distances $OP_1 : OP_2 : OP_3$ will change. But the projections of the velocities of P_1, P_2, P_3 along the tangents to the circular sections at the given points cannot change as both the reaction of the cone and the stress in the rod are at right angles to this direction. This is where the difficulty comes; our geometrical conditions require an acceleration at right angles to the direction in which the forces act.

Fig. 2.



Simple illustrations of problems of the same nature as the preceding.

In Fig. 2 two particles of mass m and m_1 are connected by a weightless rod of length a and also constrained to move along the parallel curves c and c_1 which are at the distance a. The curves are composed of straight lines with arcs of circles at one extremity. If the curves are smooth, the stress in the rod is immaterial; if no external forces act on the system, the particles may be started at A and A_1 with any common velocity but when they reach E and E_1 a difficulty is encountered, as the moment the particles move along the circular arcs their velocities are no longer equal but must be in the ratio $a + b : b$. The reactions of the curves and the stress in the rod are at right angles to the direction of motion and therefore cannot produce the required change in the velocities. We could also start the particles at D and D_1 with velocities in the ratio $a + b : b$ and when they come to E and E_1 there would be the same difficulty.

If the curves are rough and the coefficients of friction are μ and μ_1, the stress in the connecting rod while the particles are supposed to pass over the rectilinear part of the path is zero if $\frac{\mu}{m} \neq \frac{\mu_1}{m_1}$ and may assume any value when $\frac{\mu}{m} = \frac{\mu_1}{m_1}$. This latter would be an illustration of a friction problem which has, not one or two[1], but an infinite number of solutions. When the curvilinear part of the path is encountered there must be a sudden change in the ratio of the velocities due to impulsive frictional forces at E and E_1.

To determine the conditions under which this is possible we shall call T the tension in the connecting rod, V the common velocity before the impulse and V' and V_1' the velocities after the impulse. The following relations are then evident:

$$mV - \lim_{\Delta t = 0}\int_t^{t+\Delta t} \mu T\,dt = mV - \mu T = mV'.$$

$$m_1 V - \lim_{\Delta t = 0}\int_t^{t+\Delta t} \mu_1 T\,dt = m_1 V - \mu_1 T = m_1 V_1'.$$

But

$$\frac{V'}{V_1'} = \frac{a+b}{b},$$

$$\frac{mV - \mu T}{m_1 V - \mu_1 T} = \frac{m(a+b)}{m_1 b},$$

hence

$$T = \frac{\frac{a}{b}\cdot V}{\frac{\mu_1}{m_1}\left(\frac{a}{b}+1\right) - \frac{\mu}{m}}.$$

As T is essentially positive, it follows that the problem is impossible in case

$$\frac{\mu_1}{m_1}\left(\frac{a}{b}+1\right) - \frac{\mu}{m} < 0$$

Our equations do not hold the moment either of the particles is brought to rest. Hence $m_1V - \mu_1 T$ and $mV - \mu T$ are both positive

$$m_1 V - \mu_1 T = \frac{V}{m}\,\frac{m\mu_1 - \mu m_1}{\frac{\mu_1}{m_1}\left(\frac{a}{b}+1\right) - \frac{\mu}{m}}.$$

$$\therefore m\mu_1 - \mu m_1 > 0.$$

and when this last condition is satisfied both

$$mV - \mu T \quad \text{and} \quad \frac{\mu_1}{m_1}\left(\frac{a}{b}+1\right) - \frac{\mu}{m}$$

are positive.

$m\mu_1 - \mu m_1 > 0$ is therefore the condition that an impulsive stress can be determined such that the ratio $V : V_1$ is changed from unity to $a + b : b$ and this stress may be either a tension or a compression.

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[1] Appell, Mecanique Vol. II, p. 123.

For motion along the curvilinear part of the paths we have, calling s and s_1 the length of the arcs measured from E and E_1, and ω the angular velocity of the particles about O,

$$s_1 = \frac{b}{a+b}\, s, \quad \ddot{s}_1 = \frac{b}{a+b}\, \ddot{s},$$

$$\ddot{s}_1 = - \quad \omega^2 b + \frac{T}{m_1} \quad \mu_1, \quad \ddot{s} = - \quad -\omega^2 (b+a) + \frac{T}{m_1} \quad \mu.$$

The tension in the rod is consequently determined from the relation

$$\omega^2 b + \frac{T}{m_1} = \frac{\mu}{\mu_1} \frac{b}{a+b} \quad -\omega^2(b+a) + \frac{T}{m},$$

or

$$T = \frac{\omega^2 b(\mu \pm \mu_1)}{\frac{\mu}{m} \frac{b}{a+b} \mp \frac{\mu_1}{m_1}}.$$

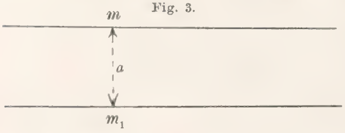
The problem has two solutions as

$$\omega^2 b + \frac{T}{m_1} = \frac{\omega^2 b \mu \left[\frac{b}{a+b} \frac{1}{m} + \frac{1}{m_1}\right]}{\frac{\mu}{m} \frac{b}{a+b} \mp \frac{\mu_1}{m_1}}$$

and these values differ according as we take the upper or lower sign, μ_1 being by supposition different from zero. One of the values of T becomes infinite in case $\frac{\mu_1}{m_1} = \frac{\mu}{m} \frac{b}{a+b}$, but a possible solution in all such cases as the one above is of course to assume an impulsive stress in the connecting rod so as to bring both the particles instantaneously to rest. This solution will not be considered.

Another illustration along the same line[1]) which may be worth examining because of its simplicity is obtained by considering the motion of two unit masses m and m_1 along parallel horizontal tubes (Fig. 3). The distance between the tubes and the length of the weightless connecting rod being a. If the coefficients of friction are μ and μ_1 and the force of gravity is the only external force acting on the particles, we at once get for the tension in the rod, if motion is to take place

Fig. 3.



$$T = g\frac{\mu_1 - \mu}{\mu_1 + \mu} \quad \text{or} \quad g\frac{\mu_1 + \mu}{\mu_1 - \mu}.$$

Suppose for definiteness $\mu > \mu_1$; then both values of T are negative, one is numerically less than g, the other greater. In the former case

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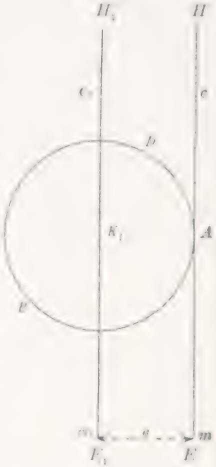
[1]) Painlevé, Leçons sur le Frottement, p. 36.

m presses against the lower surface of the tube, in the latter against the upper.

As a third simple illustration consider the two parallel curves (Fig. 4) c, composed of a straight line and a tangent circle of radius a, and c_1 a straight line parallel to the first and passing through the centre of the circle. The particles have masses m and m_1 and the coefficients of friction are μ and μ_1. If the particles are connected by a weightless rod of length a and are started from E and E_1 with a common velocity, it is easy to see that when no external forces act it is impossible for m to describe the path $EADBAH$ while m_1 describes the rectilinear segment $E_1K_1H_1$.

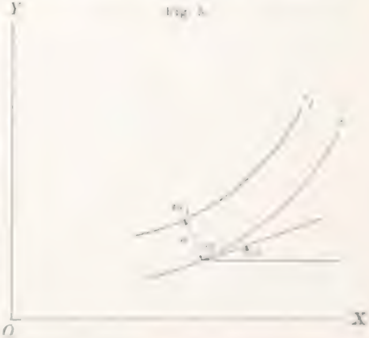
Fig. 4



Statement of the problem.

The illustrations have brought out the fact that conditions which are incompatible when there is no friction may not necessarily be so when the surfaces are rough. This is an interesting thing as we ordinarily think of the force of friction as a force which prevents, or tends to prevent, motion. Here it plays the opposite role. The illustrations also suggest the general question as to what conditions must be satisfied in order that motion may be possible when the velocity is normal to the constraining force, or perhaps it would be better to say: under what circumstances does the supposition that motion takes place lead to conditions which are compatible? I propose to treat a fairly general type of problem of this kind, viz.: *given two parallel plane curves c and c_1, the distance between the two being a (Fig. 5); two particles of mass m and m_1, connected by a weightles rod of length a, are constrained to move along c and c_1. No external forces act on the particles. Under what circumstances does the supposition that motion takes place lead to conditions which are compatible: (a) when c and c_1 are smooth, (b) when c and c_1 are rough?*

Fig. 5



Derivation of the relation between j and j_1, the tangential components of the acceleration of m and m_1.

Let (x, y) be the coordinates of m (Fig. 5) and $\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot y}{\dot x} = \tan\alpha$.

The coordinates of m_1 at the given moment are then

$$x_1 = x - a\sin\alpha, \quad y_1 = y + a\cos\alpha.$$

We get by differentiation with respect to t

$$\dot x_1 = \dot x - a\cos\alpha\,\dot\alpha, \quad \dot y_1 = \dot y - a\sin\alpha\,\dot\alpha,$$

$$\ddot x_1 = \ddot x + a\sin\alpha\,\dot\alpha^2 - a\cos\alpha\,\ddot\alpha, \quad \ddot y_1 = \ddot y - a\cos\alpha\,\dot\alpha^2 - a\sin\alpha\,\ddot\alpha.$$

$$\therefore j_1 = \ddot x_1\cos\alpha + \ddot y_1\sin\alpha = \ddot x\cos\alpha + \ddot y\sin\alpha - a\cos^2\alpha\,\ddot\alpha - a\sin^2\alpha\,\ddot\alpha = j - a\ddot\alpha.$$

Also $\dot\alpha = \frac{V}{\varrho}$, ϱ being the radius of curvature of c at the given point, V the velocity of m. Hence

$$\ddot\alpha = \frac{\varrho\dot V - V\dot\varrho}{\varrho^2} = \frac{j}{\varrho} - \frac{V}{\varrho^2}\frac{d\varrho}{ds}\dot s = \frac{j}{\varrho} - \frac{V^2}{\varrho^2}\frac{d\varrho}{ds} = \frac{j}{\varrho} - \omega^2\frac{d\varrho}{ds},$$

where $\omega = \frac{V}{\varrho}$, and s is the arc of the curve c measured in the direction of the motion of m. The relation $j_1 = j - a\ddot\alpha$ can therefore be written in the form

$$j_1 - \left(1 - \frac{a}{\varrho}\right) j = \omega^2 a\frac{d\varrho}{ds}.$$

Case (a): c and c_1 are smooth.

In this case $j_1 = j = 0$ and $\therefore \frac{d\varrho}{ds} = 0$, $\varrho = a$ constant. The curves c and c_1 are concentric circles and the velocities V and V_1 are in the ratio $\varrho : \varrho - a$.

Case (b): c and c_1 are rough, the coefficients of friction being μ and μ_1.

As before, call T the tension in the rod, V and V_1 the velocities of m and m_1; the equations of motion of the two particles are then

$$mj = -\mu\left[\frac{mV^2}{\varrho} - T\right], \quad j = -\mu\left[\omega^2\varrho - \frac{T}{m}\right],$$

$$m_1 j_1 = \mp\mu_1\left[\frac{m_1 V_1^2}{\varrho - a} + T\right], \quad j_1 = \mp\mu_1\left[\omega^2(\varrho - a) + \frac{T}{m_1}\right],$$

the upper or lower sign being taken according as $\varrho \gtrless a$; i. e. according as the velocities of the two particles have the same or opposite sense. The relation

$$j_1 - \left(1 - \frac{a}{\varrho}\right) j = \omega^2 a\frac{d\varrho}{ds}$$

becomes

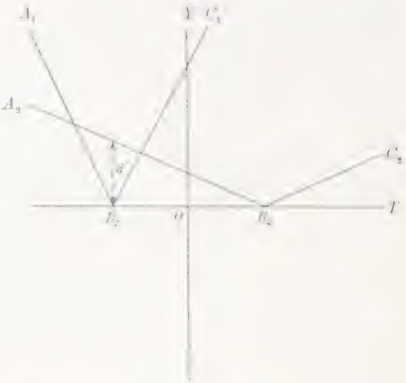
$$\mp\mu_1\left[\omega^2(\varrho - a) + \frac{T}{m_1}\right] + \left(1 - \frac{a}{\varrho}\right)\mu\left[\omega^2\varrho - \frac{T}{m}\right] = \omega^2 a\frac{d\varrho}{ds}.$$

If our conditions are compatible it must be possible to determine T so as to satisfy this equation.

To study the meaning of this condition put

$$y_1 = \left|\frac{m_1}{\mu_1}T + \omega^2\mu_1(\varrho - a)\right|, \quad y_2 = \left|\left(1 - \frac{a}{\varrho}\right)\frac{\mu}{m}T + \omega^2\varrho\mu\left(1 - \frac{a}{\varrho}\right)\right|$$

Fig. 6.



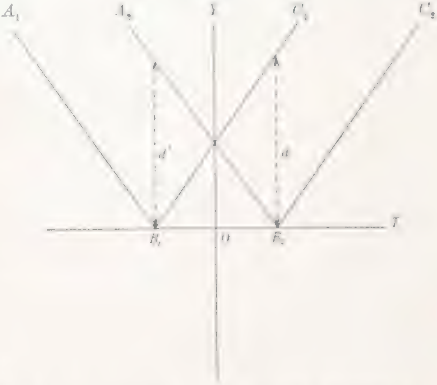
and suppose first that $\varrho > a$. The first equation is satisfied by the points on the broken line $A_1B_1C_1$ and the second by the points on $A_2B_2C_2$ (Fig. 6) and the equation which T must satisfy can be written

$$-y_1 + y_2 = \omega^2 a\frac{d\varrho}{ds}.$$

Figs. 6, 7, 8 correspond to $\frac{\mu_1}{m_1} \gtreqless \left(1 - \frac{a}{\varrho}\right)\frac{\mu}{m}$ respectively. In 6 we have two solutions, one, or none according as

$$a\omega^2\frac{d\varrho}{ds} \lesseqgtr d' - \omega^2(\varrho - a)\mu\left[\frac{\varrho - a}{\varrho}\frac{m_1}{m} + 1\right].$$

Fig. 7.



In 7 there are one, an infinite number, or no solutions according as $\left|\omega^2 a\frac{d\varrho}{ds}\right| \lesseqgtr d' - d$: in 8 there are two, one, or none according as

$$\omega^2 a\frac{d\varrho}{ds} \gtreqless -d$$

$$= -\omega^2\mu_1\left(\frac{m}{m_1}\varrho + \varrho - a\right).$$

Suppose next that $\varrho < a$: the equation in T becomes

$$y_1 - y_2 = \omega^2 a\frac{d\varrho}{ds}.$$

The different possibilities are shown in Figs. 9, 10, 11, 12, 13. The results are as follows: Fig. 9, two solutions, one, or none according as $\omega^2 a\frac{d\varrho}{ds} \gtreqless -d'$; Fig. 10, two solutions, one, or none according as

Fig. 8.

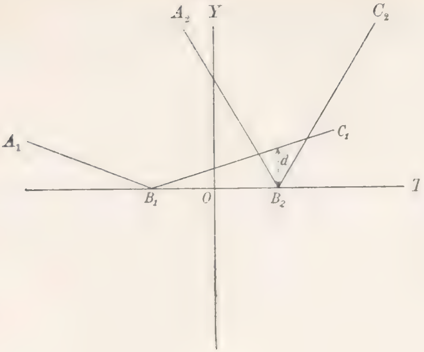


Fig. 9.

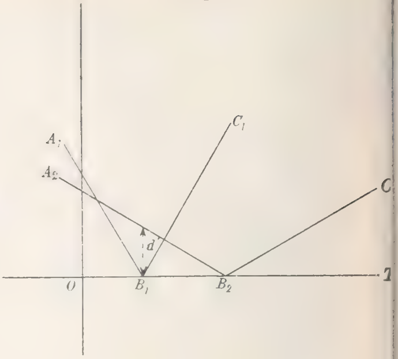


Fig. 10.

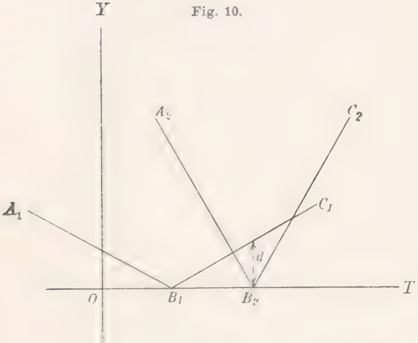


Fig. 11.

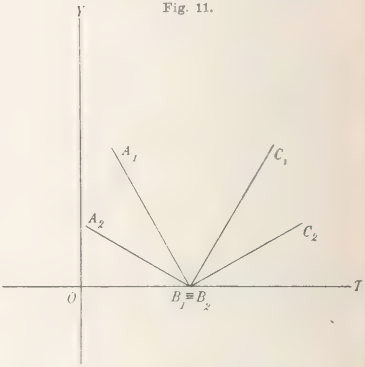


Fig. 12.

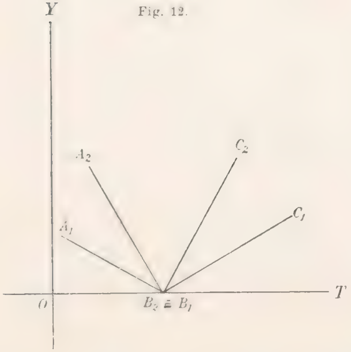
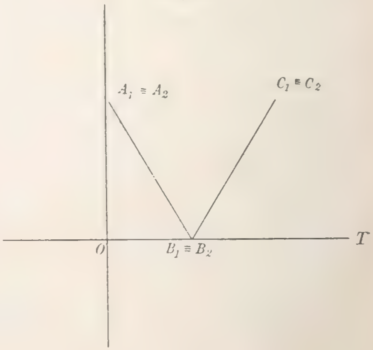


Fig. 13.

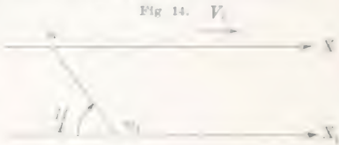


$\omega^2 a \frac{d\varrho}{ds} \lessgtr d$ (The case where the slope of the two broken lines is the same, gives the same result as for Fig. 7); Fig. 11 gives one solution if $\omega^2 a \frac{d\varrho}{ds} > 0$, otherwise none; Fig. 12 gives one solution if $\omega^2 a \frac{d\varrho}{ds} < 0$ and none if $\omega^2 a \frac{d\varrho}{ds} > 0$. The case shown in Fig. 13 is possible only when $\frac{d\varrho}{ds} = 0$, i. e. the curves must be circles and when this condition is satisfied there are an infinite number of solutions. In Figs 9 and 10 B_1 may of course be either on the right or left of B_2.

An illustration where the rod connecting the particles is not normal to the direction of motion.

We have seen in the preceding pages illustrations of friction problems which have 0, 1, 2, or an infinite number of solutions. There is no difficulty in constructing a problem which permits of a finite number of solutions greater than two and this property is not restricted to problems in which the direction of motion is normal to the constraint. As an example of this we consider the following problem.

Fig 14.



Given two unit masses, connected by a weightless rod as in Fig. 14, and constrained to move along horizontal tubes having coefficients of friction μ and μ_1: there are no external forces excepting gravity. Investigate the motion when the particles are given a common initial velocity V_0.

If we call T the tension in the rod the equations of motion of the two particles can be written

$$\ddot{x} = -\mu e\left(\frac{T}{\sqrt{2}} + g\right) + \frac{T}{\sqrt{2}},$$

$$\ddot{x}_1 = -\mu_1 e_1\left(g - \frac{T}{\sqrt{2}}\right) - \frac{T}{\sqrt{2}},$$

where e and e_1 are ± 1: the sign being determined by the fact that the product of e or e_1 and the quantity in the parenthesis must be positive. As $\ddot{x} = \ddot{x}_1$,

$$T = g\,\frac{\sqrt{2}(\mu e - \mu_1 e_1)}{2 - (\mu e + \mu_1 e_1)},$$

$$e\left(\frac{T}{\sqrt{2}} + g\right) = \frac{2g(1 - \mu_1 e_1)}{2e - (\mu + \mu_1 e e_1)},$$

$$e_1\left(g - \frac{T}{\sqrt{2}}\right) = \frac{2g(1 - \mu e)}{2e_1 - \mu_1 - \mu e e_1}.$$

The last two expressions must be positive. Let us write down their values for the different cases.

Case I: $e = 1,\ e_1 = 1.$

$$e\left(\frac{T}{\sqrt{2}} + g\right) = \frac{2g(1-\mu_1)}{2-(\mu+\mu_1)}, \quad e_1\left(g - \frac{T}{\sqrt{2}}\right) = \frac{2g(1-\mu)}{2-(\mu+\mu_1)}.$$

Case II: $e = -1,\ e_1 = -1.$

$$e\left(\frac{T}{\sqrt{2}} + g\right) = \frac{2g(1+\mu_1)}{2-(\mu+\mu_1)}, \quad e_1\left(g - \frac{T}{\sqrt{2}}\right) = \frac{2g(1+\mu)}{-2-(\mu+\mu_1)}.$$

Case III: $e = -1,\ e_1 = 1.$

$$e\left(\frac{T}{\sqrt{2}} + g\right) = \frac{2g(1-\mu_1)}{-2-(\mu-\mu_1)}, \quad e_1\left(g - \frac{T}{\sqrt{2}}\right) = \frac{2g(1+\mu)}{2-(\mu_1-\mu)}.$$

Case IV: $e = 1,\ e_1 = -1.$

$$e\left(\frac{T}{\sqrt{2}} + g\right) = \frac{2g(1+\mu)}{2-(\mu-\mu_1)}, \quad e_1\left(g - \frac{T}{\sqrt{2}}\right) = \frac{2g(1-\mu)}{-2-(\mu_1-\mu)}.$$

An examination of these different cases shows that II is impossible but that it is possible to choose the coefficients of friction so as to admit the others: e. g. $\mu = 3,\ \mu_1 = 4$ gives the following possibilities

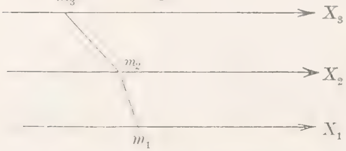
(1) $$T = \frac{\sqrt{2}}{5}g, \quad \ddot{x} = \ddot{x}_1 = -\tfrac{17}{5}g,$$

(2) $$T = -7\sqrt{2}\,g, \quad \ddot{x} = \ddot{x}_1 = -25\,g,$$

(3) $$T = \tfrac{7}{3}\sqrt{2}\,g, \quad \ddot{x} = \ddot{x}_1 = -\tfrac{23}{3}g.$$

The conditions of the problem are also satisfied (being μ and μ_1 are both greater than $\frac{\pi}{4}$) by assuming that both the particles are instantaneously brought to rest by an impulsive stress in the rod. The impulse of this impulsive stress which is required in order to bring m to rest is in general different from that required to bring m_1 to rest, but the point is that the moment one of the particles is brought to rest any additional impulsive stress in the rod has no effect on the particle which is already at rest.

Fig. 15.



The results of the illustration may be summarized briefly by saying that there are four solutions which satisfy the conditions of the problem, and there are three distinct motions which satisfy the conditions of the problem.

A problem with a still larger number of solutions may be obtained by considering the motion of three or more particles along rough horizontal tubes as in Fig. 15.

It may be worth while to call especial attention to the fact that throughout this discusssion we have assumed that we were dealing with a rigid system and the problem has simply been to see what the equations of motion led us to in case the constraints were normal to the velocity. Should we take into account the elasticity of the material, we would have a different problem and we would no longer be dealing with a problem in the mechanics of a rigid system.[1])

University of Michigan, June 1912.

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[1]) Appell, Mecanique, 3^e edition, Vol. II, p. 127.

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THE STABILITY OF RESIDUAL MAGNETISM.

By N. H. Williams.

A. Theoretical.

I. Analogies between Elastic and Magnetic Phenomena.

THE magnetization of iron under the influence of a magnetizing field is often considered as a phenomenon analogous to the distortion of an elastic body under stress. The elastic strain theory has been built upon this analogy. The intensity of a magnetizing field at a point is defined as the force per unit pole at the point. It is a vector quantity and in a homogeneous medium is a continuous function, hence it is permissible to think of the flux of the vector per unit area perpendicular to the vector as a measure of the field intensity; thus it becomes analogous to stress in a body which is in a condition of strain. The substance in which the magnetic field is established undergoes a change which is also a vector point function and its flux per unit area is called the magnetic induction at a point, and is analogous to the strain in a body under stress. The ratio of the magnetic induction to the intensity of the magnetizing field is called the permeability of the medium and is analogous to the reciprocal of an elastic modulus.

The deformation of an elastic body occurs in approximate accordance with Hooke's law; but beside this approximate proportionality of stress and strain, other effects are observed such as elastic after-effect or hysteresis, plastic yield, permanent set due to repeated application of the load, and the increase of elastic limit with over-strain.

A magnetic phenomenon which is analogous to elastic after-effect may be observed when a demagnetizing field is applied to magnetized iron or steel. After the first partial demagnetization due to the field, there is a further slight demagnetization which is too slow to produce an appreciable effect upon a ballistic galvanometer. There is a similar time lag when the field is in such a direction as to increase the induction, and the effect is much more apparent in soft iron than in steel. These facts have been mentioned by Lord Rayleigh[1] and by Ewing.[2]

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[1] Phil. Mag., March, 1887.

[2] Phil. Trans., 1885, p. 569.

Plastic yield and permanent set are analogous to the magnetic phenomena resulting from any change of magnetic induction exceeding a limit beyond which the specimen fails to return to its former state upon being removed from the field.

It was first shown by Fromme[1] and afterward by Ewing[2] that repeated applications of a magnetizing field produce a higher residual induction than a single application. This is analogous to the fact that repeated application of the load upon an elastic body may produce a permanent set provided the stress approaches the elastic limit.

II. *Definition of Terms.*

It will be convenient to use the term elastic limit as applying to magnetic phenomena and meaning the maximum intensity of field which may be applied to a specimen without a permanent change of its induction. A *partial* demagnetization will be designated merely as a demagnetization and the word recovery is here used to represent the change of flux which occurs when the demagnetizing field is removed.

III. *Purpose of the Work.*

The purpose of this paper is to present some experimental work which suggests further analogies between elastic and magnetic phenomena and especially to determine for different kinds of iron and steel the relation between the recovery and the maximum residual flux.

IV. *The Scope of the Work.*

The experiments show an approximate proportionality between magnetic stress and strain and that the limit beyond which the iron fails to regain its former condition after partial demagnetization may under favorable circumstances reach a variation of magnetic induction as high as 2,000 lines per sq. cm.

It is shown that for hard steel the hysteresis, for changes within the elastic limit, is quite small and the analogy between magnetic hysteresis and mechanical hysteresis is made evident by comparing this work with work recently done upon the elastic properties of steel tubes.

The effect of repeated application of a *demagnetizing* field is found to be similar to that found by Fromme and Ewing for a *magnetizing* field, *i. e.*, many repetitions of a magnetic stress, which is equal to the elastic limit, produce a change of induction that is equivalent to a permanent set.

An analogy is found for the increase of elastic limit with over-strain.

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[1] Pogg. Ann., 1875; Wied. Ann., IV., 1878.

[2] Phil. Trans., 1885, p. 570.

B. Experimental.

I. Madelung's Experiment.

Madelung[1] has published an account of an excellent piece of work on the magnetizing effects of high frequency currents. His method involved the use of the Braun tube in plotting variation of magnetic flux in steel wire with variation of field intensity caused by the discharge current from a condenser. Fig. 1 is a copy of one of his curves showing the demagnetizing effect of an aperiodic discharge. He plots what he calls the maximum hysteresis curve, *i. e.*, the curve resulting from carrying the magnetizing field to a very high value and then reducing it. It will be observed that the curve due to the discharge lies outside the maximum curve. This is due to the effect of eddy currents in the steel, which set up a counter magnetizing field and cause a shift in phase of the actual magnetizing field. The curve obtained from the instrument gives the relation of the magnetic induction to the magnetizing field as it would be if not modified by eddy currents. The three curves of Fig. 1 were taken with wires of different size, the curve taken with small wire being the lowest one. Since the eddy current effects decrease with diminished size of wire, this lowest curve would be expected to conform more nearly to the maximum hysteresis curve than the others, as it does.

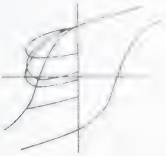

Fig. 1.

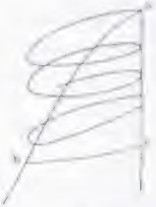

Fig. 1a.

Fig. 1*a* shows the effect of repeated application of the demagnetizing impulse. The curve falls to lower values of the induction as the impulse is repeated and finally reaches a stationary position where it becomes a closed curve. However, it never falls to a position where its extreme point lies on the hysteresis curve, but the point remains outside the area of the curve. Madelung points out that this is an error due to the eddy currents and suggests that if the current were of long duration and its time rate of variation at all times small, the curve traced would be the maximum hysteresis curve down to the point *b* and from there it would be the line *bC*. This is undoubtedly true for the first application of the demagnetizing field but the question arises: what would be the result of many repetitions of the demagnetizing impulse? A part of the experimental work herein described deals with this question. An effort has been made to reduce the errors due to eddy currents and end effects to a negligible magnitude and furthermore to deal with the whole matter in a quantitative way so that we may know to just what extent demagnetization is followed by recovery and how much hysteresis occurs.

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[1] Annalen der Physik, 17, 1905, p. 861.

II. *General Method.*

The method adopted in the experimental work was an oscillographic method devised by the author and so far as he is aware it has not been used elsewhere. The specimen to be studied was of steel wire in the form of a ring wound with primary and secondary coils. The secondary of a transformer operating on a sixty cycle current was connected in series with a storage battery and adjustment made so that the maximum value of the alternating E.M.F. was equal to the storage battery E.M.F. This gave a unidirectional pulsating current fluctuating between the limits of zero and a maximum value. This current was made to pass through the primary coil on the magnetized core in such a direction as to produce a demagnetizing field. A portion of this current operated one loop or vibrator of the oscillograph while the other loop was connected to the secondary consisting of a few turns and also wound on the steel core. The inductance in the secondary was negligible and hence the ordinate of the curve obtained by the oscillograph from this coil is proportional to the rate of flux variation while the ordinates of the current curve represent the instantaneous value of the magnetizing field. This latter statement is only true provided the results of eddy currents and end effects are negligible.

Proper calibration is necessary in order to obtain actual values of field intensity and rate of flux variation from these curves.

In order to obtain the curve whose ordinates represent the change of flux due to the demagnetizing field, the curve whose ordinates represent the time rate of change of flux is integrated graphically, the process being as follows: The curves as photographed by the oscillograph are transferred by means of a pantagraph to cross-section paper, the copy being drawn to double scale. The paper used was divided into twentieths of an inch. At the instant when the ordinate of the curve representing rate of flux variation is zero and increasing in the positive direction, the flux is a maximum. Directly above this point *a*, Fig. 2, at the top of the sheet a line *bc* is drawn across the quarter inch in which this point lies parallel to the axis of abscissæ. Across the next quarter inch a line *cd* is drawn from the extremity of the preceding line and its slope is made equal to the negative of the ordinate *mn* of the curve representing time rate of flux variation at the middle point of this quarter inch. Where this line ends, another begins and crosses the next quarter inch with a negative slope equal to the ordinate of the flux rate curve at the middle of this space. The process is continued until the curve for the cycle is completed. If the work is carefully done the ordinate of the curve, measured downward, represents the flux variation with considerable accuracy. If the ordinary hysteresis curve is wanted an alternating current is used instead of the unidirectional pulsating current. The values of *H*, as computed from the ordinates of the current curve are plotted as abscissae and the values of the flux or the induction *B*, computed from the flux curve, are plotted as ordinates. Thus we have the hysteresis curve when the alternating current is used and the demagnetization and recovery curve when the pulsating current is used. A hysteresis curve obtained by this method is shown on Fig. 7, the specimen being piano wire in its normal condition.

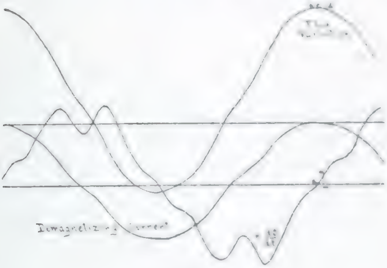
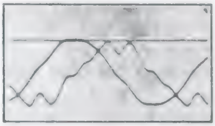


Fig. 2.



Oscillogram. The smooth curve represents the unidirectional primary current; the irregular curve the secondary current.

Fig. 2*a*.

The method has several advantages over the ballistic method for work involving small changes of *B* and *H*. It is possible to determine as many points on the curve as desired even when the curve represents very small changes. The apparatus is quite sensitive to small changes of flux while for large changes it is only necessary to reduce the number of secondary turns and put resistance into the circuit. For example, one curve was plotted in which the flux variation was only 114 lines per sq. cm. and it showed all the characteristics of the curves representing much larger changes; on the other hand hysteresis curves representing changes of magnetic induction of more than 30,000 lines per sq. cm. are easily obtained. The method also insures conditions giving a closed curve for the cycle since the impulse is repeated a large number of times before the photograph is taken and the cycle is traversed more than 1,000 times during the exposure of the sensitive paper.

The remaining details of the method were different for different specimens and will be mentioned in connection with the experiments to which they apply.

III. *Calibration.*

A ballistic galvanometer was used for a part of the work. It was calibrated by means of a current inductor in the usual way.

One mm. deflection was found to correspond to a flux variation of 32.6 lines of induction.

The same current inductor was also used in calibrating the oscillograph. A wire was used for the secondary coil of such a size that its resistance per turn was the same as that of the secondaries of the steel cores. This secondary was connected to the same loop of the oscillograph that was afterward used for the secondaries of the steel cores. The maximum flux through the coils was computed from the same formula as that used in calibrating the galvanometer. Calibration curves were taken with 25, 50, 75 and 125 turns respectively in the secondary. The unidirectional current was used here as in the work upon the steel cores. The flux curve in each case was obtained by graphic integration as previously described. The maximum ordinate of this curve corresponds to the flux as computed from the maximum current. The value in terms of lines of induction of one division of the cross-section paper is thus determined, *e. g.*, with 25 turns one division corresponds to 22.9 lines of induction. Since the maximum flux variation is used in this determination, it is evident that it must be computed from the maximum current rather than from the effective value.

Since a direct current instrument having its field furnished by a permanent magnet indicates the mean current flowing through it and since the mean value of an alternating current furnished by a transformer is zero, a direct current ammeter reads one half the maximum current when the current is unidirectional and fluctuates between zero and a maximum value. The instrument used was a Weston ammeter reading directly to .02 of an ampere and having large divisions so that one could estimate to .005 ampere with accuracy.

As an illustration of the accuracy of the process, two calibration curves for a secondary of 75 turns were taken at different times, one about two weeks later than the other and under conditions giving very different wave forms. The flux value of one division of the paper was 8.104 in one case and 8.080 in the other, showing a difference of two tenths of one per cent.

A separate calibration was made for plotting the hysteresis curves of the specimens. This became desirable since a secondary of only one turn was used and some resistance was inserted to reduce the induced current in the secondary circuit to a value that could be used with the oscillograph.

The magnetic flux per division of the cross-section paper as given by the calibration curves was as follows, the numbers in the first column being the number of secondary turns.

125 turns.	6.65 lines.
75	8.09
50	11.94
25	22.90

IV. Experiments with Specimen A.

The first specimen to be studied was of piano wire 1.03 mm. in diameter. It was in the form of a ring of mean circumference 25 cm. and cross-sectional area 1.4 sq. cm. A gap was cut in the ring about one cm. long and a plug of soft iron wire was made to fit it. The ring was heated to a red heat in an electric furnace and then plunged into cold water. The core was next insulated with tape and a secondary coil consisting of five sections of twenty-five turns each was wound upon it. The primary of 350 turns was wound outside the secondary.

The gap was made in the ring in order that it might be possible to determine at any time the residual flux in the core. This determination was made by means of the galvanometer and test coil previously calibrated. With the coil encircling the ring on the side opposite the gap, the plug was removed and the throw of the galvanometer noted. The coil was then quickly slipped off the ring and a second deflection observed. Since the calibration showed 32.62 lines of induction per mm. deflection, the total flux was found by multiplying 32.62 by the sum of the two deflections.

A series of photographs was taken for the glass-hard steel with maximum demagnetizing field ranging from 1.65 gauss to 34.5 gauss. The variation of induction in the first case was 114 lines of induction per sq. cm. while in the latter it was 1,960. In this latter case the remanence was 140 lines per sq. cm. and the application of the current caused a reversal of the magnetism, producing a negative induction of 1,820 lines per sq. cm. As the current was reduced to zero the magnetic flux returned to its former positive value. With somewhat smaller demagnetizing currents the magnetism was not reversed and the recovery on reducing the field to zero was from a value of the induction which was still positive to a higher positive value. For example, when the current was 1.56 amperes, giving a field of 27.4 gauss, the induction fell from 1630 to 60 and returned to 1,630 when the current became zero. The magnitude of these recoveries is appreciated when they are compared with the maximum residual magnetism observed in the specimen. This latter value was 8,036 lines per sq. cm.; the recovery of 1,960 is therefore 24.5 per cent. of the maximum residual induction.

The curves for this specimen were taken in order beginning with small demagnetizing current. The first application of the current produced a certain amount of permanent demagnetization. After many applications of the current the magnetism reached a stable condition in which the recovery was just equal to the demagnetization. Then the photograph was taken. Next a larger current was applied which produced a further permanent demagnetization and a new condition of stability. Then another photograph was taken. The method of determining the residual flux after each photograph was taken has been described. The removal of the plug before each observation introduced a demagnetizing field due to the ends and caused a marked decrease of the residual magnetism when the flux was at its high values. This effect decreases as lower values of the remanence are reached and vanishes when the flux becomes zero. These curves were plotted with magnetic flux as ordinates rather than the flux density or induction while the figures quoted above are values of flux density. The area of the specimen was 1.4 sq. cm. hence to obtain variation of induction from the curves the ordinates should be divided by 1.4.

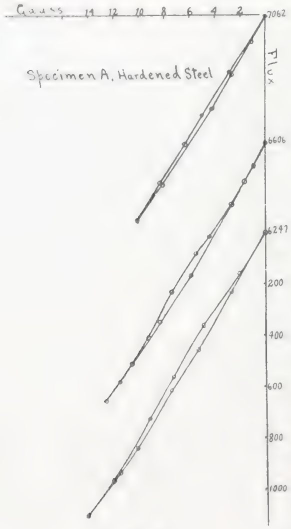


Fig. 3.

Figs. 2 and 3 are two of the series obtained from this specimen. The curves were all transferred to a single sheet (see Fig. 4) and properly placed with reference to the axes according to the residual flux determined after each was taken.

Data for Demag. and Recovery Curves of Specimen A.

Current.	No. of Secondary Turns.	Def. on Removing Plug.	Def. Due to Remaining Flux.	Remanent Flux.	Maximum H.	Gauss per Division.
.094	125	24.0	207	7,535	1.6537	.0438
.48	75	21.0	198	7,144	8.4446	.1206
.576	50	20.5	196	7,062	10.1336	.1842
.72	50	19.5	183	6,606	12.667	.2112
.80	50	18.5	173	6,247	14.074	.2346
.92	50	17.5	159	5,757	16.185	.234
1.00	25	16.0	146	5,284	17.593	.3824
1.18	25	14.0	122	4,436	20.76	.3844
1.32	25	12.0	99	3,621	23.222	.3776
1.40	25	10.0	88	3,197	24.63	.4561
1.56	25	7.0	63	2,283	27.445	.524
1.68	25	6.0	49	1,794	29.556	.5185
1.80	25	3.5	32	1,158	31.667	.5278
1.96	25	0	6	197	34.482	.6000

Data for Demagnetization and Recovery Curves Derived from the Current Curve and the Flux Variation Curve.

H = field intensity. ϕ = magnetic flux.

H	ϕ	H	ϕ	H	ϕ	H	ϕ
.24	22.3	.75	70	1.105	107	1.02	95
.48	49.9	2.08	168	2.763	233	2.95	238
.85	81.8	3.98	308	4.973	399	4.50	368
1.40	136.0	5.18	401	6.263	513	5.60	440
1.65	159.0	6.51	522	8.289	668	7.50	584
1.36	133.0	8.44	656	10.133	812	11.55	938
.70	68.0	6.93	559	8.289	668	12.67	1015
.48	46.2	5.43	437	6.080	501	10.60	862
		3.74	308	4.140	370	7.50	582
		2.65	219	2.763	239	5.58	438
		1.03	93			4.50	365
						1.89	155
2.111	167	1.872	191	1.912	160	1.53	130
4.93	370	4.446	406	4.589	389	3.84	321
7.27	573	7.956	668	8.413	664	7.88	641
9.15	740	10.070	859	10.707	870	11.92	916
11.96	979	13.104	1134	14.149	1164	14.61	1214
14.07	1110	16.180	1432	16.443	1374	18.45	1540
12.43	1014	13.104	1183	17.59	1466	20.76	1740
10.09	847	9.600	919	16.44	1374	18.07	1557
7.51	621	6.790	692	14.53	1213	15.00	1305
5.40	465	4.446	441	11.472	962	11.15	985
2.81	239	14.04	167	7.839	733	7.88	733
				5.162	504	4.23	378
				1.912	183	.87	114

H	ϕ	H	ϕ	H	ϕ
3.77	321	1.82	137	1.57	160
8.31	664	4.11	343	4.19	400
12.46	1054	8.66	710	9.43	836
15.86	1374	12.77	1053	14.67	1150
20.01	1740	16.41	1397	18.34	1534
23.26	1992	20.75	1786	23.31	1946
20.39	1786	24.86	2038	27.45	2198
16.61	1511	21.89	1866	23.80	1969
12.08	1145	18.24	1580	19.38	1649
8.68	847	13.68	1191	18.34	1237
4.91	458	9.12	893	10.18	962
		5.93	470	5.67	481
		1.82	149	1.05	137
1.80	183	2.64	251	4.5	378
4.67	492	7.38	618	9.9	801
10.37	939	13.20	1100	16.8	1340
16.07	1340	18.50	1511	22.8	1763
20.48	1800	23.75	2015	27.3	2255
25.93	2233	29.46	2427	34.5	2748
29.55	2496	31.67	2611	27.3	2358
25.93	2233	30.08	2473	22.2	1981
21.26	1855	26.64	2267	15.3	1488
15.54	1397	21.00	1855	9.9	1008
10.89	1031	14.98	1397	4.5	423
5.70	527	10.55	973		
1.39	137	4.22	424		

It will be observed that the extreme points of the demagnetization curves lie on a smooth curve. The determination of the relation of this curve to the maximum hysteresis curve was next undertaken. This portion of the hysteresis curve was determined ballistically and was found to lie outside the locus of the extreme points and as would be expected deviated widely from it near the upper end. This deviation was due to the end effects established when the plug was removed as previously explained. A second ballistic curve was traced under the same conditions that existed when the work on the oscillographic curves began, *i. e.*, the core was highly magnetized, then the plug was removed and replaced, thus the remanence was brought down to the same value at which it stood when the oscillographic work began. This second ballistic curve follows quite closely the contour of the locus of the extreme point of the demagnetization curve as is seen on Fig. 4. However, it lies outside of this locus as does the maximum hysteresis curve.

That the relation of these two curves is not due to error is shown by further work by the ballistic method. The specimen was first highly magnetized, then a certain demagnetizing current applied sufficient to bring the residual magnetism down to the region in which the oscillographic curves were obtained. Conditions would then be represented by some point on the maximum hysteresis curve. From this point the curve of recovery was traced ballistically, as many points being determined as seemed expedient. This curve was found to differ from that obtained by the oscillograph for the same demagnetizing current both in position and slope. The conditions differed in the two cases only in this: the ballistic curve was obtained the first time the steel was put through the cycle while the oscillographic curve was taken after several thousand repetitions of the cycle. It was next found that after opening and closing the circuit through the primary, a second ballistic curve approached more nearly the oscillographic curve. When the key in the primary circuit was opened and closed about a hundred times and the ballistic curve then traced, it was found that its extreme point fell almost exactly upon the locus of the extreme points of the oscillographic curve, and furthermore the slope and shape of the new curve agreed with those of the oscillographic curve. A number of cycles of demagnetization and recovery were traced ballistically. Some of them are shown on Fig. 4. in their relation to the other curves.

In plotting the hysteresis curve by the ballistic method, an interesting fact appeared regarding end effects. The residual induction after the application of a strong magnetizing field and before the plug was taken out was about 8,000 lines of induction per sq. cm. When the plug was removed the end effects caused it to fall to a value of 4,940 lines per sq. cm. The core was then remagnetized with the plug in place and a demagnetizing current applied. This current might be of any strength provided it did not reduce the induction below 4,940. The circuit carrying this current was then opened and a certain recovery ensued. Then the plug was removed and it was found that the residual induction was again 4,940. This constancy of the residual

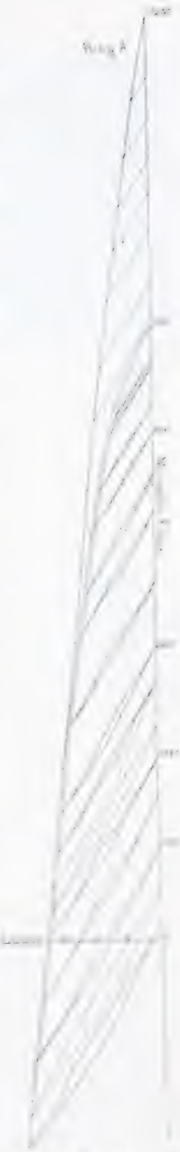


Fig. 4

magnetism on open magnetic circuit persisted until a demagnetizing filed of 12.8 gauss had been applied and then suddenly the residual magnetism, with the magnetic circuit open, fell to lower values. The point at which it breaks down is the point where the recovery on removing this demagnetizing field of 12.8 gauss is equal to the demagnetization due to the ends when the plug is removed. Fig. 5 shows these relations in graphic form. Curve *A* represents the maximum hysteresis curve. The inercept between *A* and *B* is the recovery upon breaking the circuit. The intercept between *B* and *C* is the demagnetization following the removal of the plug. At the right of point *P* the demagnetization due to the ends exceeds the recovery while at the left of *P*, the recovery is greater. Intercept *PQ* represents both demagnetization due to the ends and recovery after removal of the field and it is at this point that the residual magnetism with the gap open ceases to be constant. The interpretation of this horizontal portion of curve *C* seems to be this: that there is a welldefined upper limit beyond which the remanence of a steel core with exposed ends cannot be made to pass.

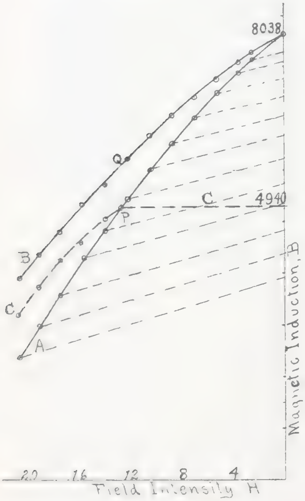


Fig. 5.

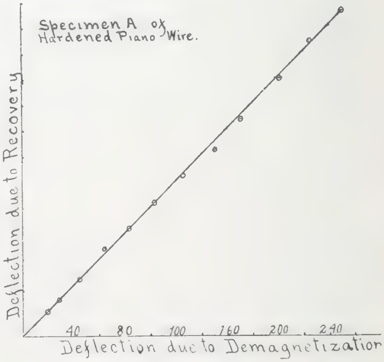


Fig. 6.

As has been mentioned, the recovery curves for a given specimen are practically parallel. For very hard steel the portion of the hysteresis curve with which we are concerned does not differ much from a straight line. If it were exactly a straight line, it would follow that the recovery would be strictly proportional to the difference between the maximum residual flux and the residual flux corresponding to the recovery curve. The line of Fig. 6 is presented to show how nearly this statement of proportionality approximates the truth.

In designing magnetic circuits, so much attention is usually given to reducing the magnetic reluctance of the joints that it seems desirable to find out if the reluctance of the joints had any influence upon the preceding experiments. A demagnetization and recovery curve was obtained by the oscillograph with the plug out of the circuit; thus the magnetic circuit contained an air gap about one centimeter long. The only marked difference between this curve and the others was a decrease in slope amounting to about thirteen per cent. This would lead to the conclusion that when the gap was filled with soft iron, the effect of the joints, upon the form or slope of the demagnetization and recovery curves would be insignificant.

V. *Experiments with Specimen B.*

The second specimen studied was a ring of no. 4 piano wire in its normal condition, *i. e.*, without heat treatment. It consisted of 1,215 wires each having a cross-sectional area of .0012 sq. cm., the aggregate area being 1.458 sq. cm. The mean length of the ring was 32.67 cm. The secondary coil consisted of five sections of 25 turns each. Outside the secondary, the primary coil of 350 turns was wound. The intensity of magnetic field within the metal was $\frac{4\pi\cdot 350\, i}{10\cdot 32.67}$ gauss, i being the current in amperes. When a test coil of fourteen turns was used, one mm. deflection of the ballistic galvanometer indicated a change of flux of 32.62 lines or a change of flux density (B) of $32.62/1.458 = 22.37$. In dealing with softer specimens, which are much more susceptible to end effects, it seemed best to try to do the work without making any air gap in the magnetic circuit, hence this ring and two others of soft iron which were used were not cut.

Plate 7 shows the hysteresis curve by the oscillographic method when the maximum field intensity is about 62 gauss. The residual induction is seen to be 13,665. As a check upon this curve a determination of the residual inductance was made by the ballistic method using the same maximum magnetizing field and it was found to be 13,670. Such close agreement is probably the result of a fortunate observation as it is beyond the limits of accuracy that could reasonably be expected.

A series of demagnetization and recovery curves was obtained for this specimen by the oscillographic method. They differ in slope from those of the former series but otherwise show very little difference.

Since there was no opening in the magnetic circuit, the method used in the former case of determining the position of the curves with reference to each other and to the residual induction could not be employed. In the case of ring A the curves were placed each with its highest point at the point representing the residual flux actually measured after the curve was photographed and the curve through their extreme lower points was drawn. The results of that work would warrant us in taking the reverse process this time. The locus of the extreme points lies within the maximum hysteresis curve. A hysteresis curve that approaches the maximum hysteresis curve very closely can be obtained by the ballistic method but just where the locus of the extreme point lies with reference to this curve is a matter to be determined. This was done by means of a Koepsel permeameter. A field of 134 gauss was applied to a specimen of this wire, this value of field intensity being chosen in order to reproduce the conditions under which the ballistic curve was taken. The field was then reduced to zero leaving a remanence of 15,400 lines per sq. cm. A small demagnetizing field was next applied and the value of the flux density, B, observed from the instrument. Then the circuit through the magnetizing coil was opened and closed 50 times and another reading of B taken. The key was then opened and closed 50 times more and the value of B again observed. Next a larger demagnetizing field was applied and the process repeated. Fourteen points on this portion of the hysteresis curve were thus found and also fourteen on a curve lying within the hysteresis curve. This latter curve, it was assumed would represent the locus of the extreme points of the recovery curves provided the hysteresis curve were accurate. In using the permeameter it is necessary to correct the observeed curve by adding to its abscissæ the abscissæ of a shearing curve. No shearing curve

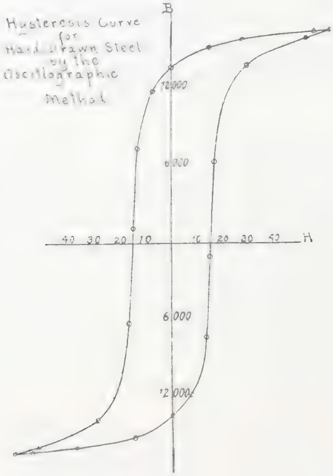


Fig. 7.

was at hand that could be assumed to be accurate for this specimen and so the ballistic curve shown in Fig. 8 was used. The curve of the extreme points of the recovery curves would lie within this hysteresis curve and it was assumed that the difference in ordinates of these two curves would be the same as the difference between the ordinates of the corresponding curves given by the permeameter. In this way the locus of the extreme points was determined. Both these curves are shown in Fig. 8. The oscillographic curves of demagnetization and recovery were now reduced to the same scale and placed upon the sheet in such a way that their extreme points fell upon the curve which lies within the hysteresis curve. The maximum recovery observed was 1,200 lines per sq. cm. and occurred in the region of zero residual flux while the maximum remanence was 14,900, thus the recovery is about 8 per cent. of the maximum remanence. The maximum recovery for the hardest steel was 24.5 per cent. of the maximum remanence. This large difference is due both to a smaller recovery and to a larger remanence in the second case.

Attention has been drawn to a number of analogies between elastic and magnetic phenomena. Bridgman[1] has recently done a piece of work on "The Collapse of Thick Cylinders under High Hydrostatic Pressure" that has furnished curves bearing a striking resemblance to the curves of Figs. 4 and 8. Fig. 13 is a copy of one of Bridgman's figures inverted for the purpose of comparison with the magnetic curves. The specimen is a hollow steel cylinder and the strain is measured by the change of internal volume. Three points of resemblance are to be observed: (1) The hysteresis, (2) the increase of elastic limit with overstrain, (3) the effect of repeated application of the same stress as shown near the bottom of the figure. The actual change of elastic limit with over-strain is of course much less than the apparent change, a large part of that appearing in the figure being due to change of dimensions of the specimen.

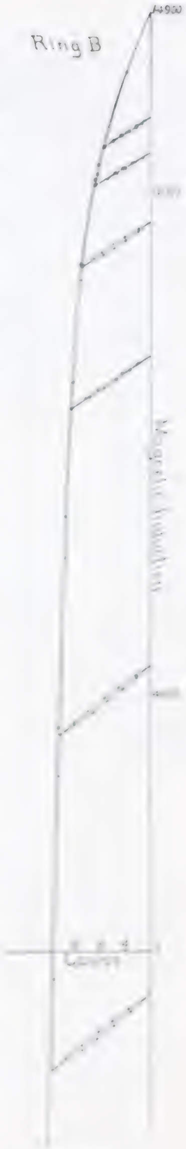


Fig. 8.

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[1] Phys. Rev., January, 1912, p. 1.

The demagnetization and recovery discussed in the previous pages have taken place under conditions such that the metal was carried during each cycle to the limit of its elasticity for the particular remanence to which it returned. For lower values of remanence the elastic limit is higher. The question would naturally arise as to the effect of a demagnetizing field that did not approach the elastic limit. The shape of the curves already taken would lead one to expect to find the change of flux due to a given demagnetizing field very nearly independent of the residual flux. This would follow from the fact that magnetic stress and strain have been found approximately proportional. Lord Rayleigh[1] has shown that the change of induction resulting from a very small change of field intensity is independent of the magnetic state of the iron, or to quote his conclusion, "That the value of the susceptibility to small changes of force is approximately independent of the initial conditions as regards force and magnetization until the region of saturation is approached." He was dealing with field strengths ranging from .04 gauss to .000018 gauss. The oscillographic method was applied to this problem, using a much larger demagnetizing field. The results may be made clear by referring to two curves produced by the same field intensity. In one case the residual flux density was 12650 and in the other 500 lines of induction per sq. cm. The maximum field intensity was 7.27 gauss. The reduction in flux density from the higher remanence was 440 while it was 475 from the lower. This represents a difference of about 8 per cent., and means that the susceptibility to small changes of field intensity increases as the remanence decreases. Similar results were obtained by the ballistic method.

Data for Demagnetization and Recovery Curves of Specimen B.

$$\text{Max. } H = \frac{4\pi \cdot 350 \cdot i}{10 \cdot 32.67} = 13.46$$

i being the current.

Current.	No. Turns in Secondary.	Maximum H.	Gauss per Division.
.27	1.25	3.6342	.0512
.54	.50	7.2684	.1191
.54	.50	7.2684	.1483
.64	.75	8.616	.1325
.79	.50	10.6334	.1715
.92	.50	12.385	.1567
1.06	.50	14.2676	.1719
1.17	.25	15.75	.2045
1.13	.50	15.21	.1925

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[1] Phil. Mag., 23, 1887, p. 225.

Data for Demagnetization and Recovery Curves of Specimen B.

Value of one division in gausses taken from preceding table. Value of one division in lines of magnetic induction determined by the calibration. H = field intensity. B = variation of induction.

H	B	H	B	H	B
.427	25	.476	23	.89	41
1.254	77	1.071	82	2.14	139
2.330	160	2.50	156	3.40	213
3.020	212	3.69	225	4.59	286
3.630	265	4.78	303	5.77	368
2.690	219	5.97	368	7.27	475
1.970	164	7.27	440	5.77	384
.947	77	5.35	327	4.37	303
.358	32	3.94	254	3.18	238
		2.91	180	2.07	156
		1.67	98	1.03	74
		.70	35		
1.06	58	1.95	110	1.88	122
2.78	150	4.13	240	4.39	262
4.19	228	5.90	344	6.11	393
5.51	322	7.70	483	8.15	540
7.50	439	9.06	573	10.18	667
8.62	511	10.63	704	12.38	835
6.62	408	7.35	532	9.40	647
5.24	319	5.64	426	7.13	516
3.84	233	3.93	311	5.33	377
2.32	139	1.79	172	3.13	221
.99	60			1.25	82
2.15	156	1.02	71	1.16	49
4.99	328	3.58	235	1.93	98
7.23	491	6.14	416	5.77	266
9.92	695	8.28	585	8.08	385
12.04	880	11.04	806	9.72	491
14.27	1064	13.50	1015	12.80	638
12.04	901	15.75	1194	15.21	761
8.94	737	10.84	871	12.03	610
6.88	573	8.59	699	8.18	454
4.30	360	5.83	487	6.45	361
2.06	180	3.07	267	1.35	74

VI. *Experiments with Specimen C.*

Specimen C is a ring of transformer iron of mean length 37.93 cm. and cross-sectional area 2.07 sq. cm. The primary coil has 325 turns, hence the magnetizing field is

$$\frac{4\pi \cdot 325\, i}{10 \cdot 37.93} = 10.767\, i.$$

One division deflection of the galvanometer corresponds to a change of induction of 15.76 lines per sq. cm. when there are fourteen turns in the test coil.

When the demagnetization and recovery curves for this specimen were plotted it was apparent that they were somewhat different from those obtained from the steel specimens. The hysteresis seemed much larger. A part of this was due to the scale to which the curves were plotted, but when allowance was made for that, they still seemed to have an abnormally large area. This peculiarity of the curves for this specimen is undoubtedly due to eddy currents in the iron. These currents are in opposition to the time rate of change of magnetic flux and hence lag behind the magnetizing current by an amount somewhat in excess of 90°. They produce a magnetic field one component of which is in direct opposition to the applied field while the other reacts to cause a lag in phase of the field behind the current. Since the curves are plotted on the assumption that the applied field is proportional to the ordinate of the current curve, it is evident that the lag of induction behind the field intensity as given by the curves would be greater than the true hysteretic lag. When there is hysteresis without eddy currents the induction shows a lag behind the current at some points along the curve but none whatever at points of maximum or minimum field intensity. A difference of phase between the two curves on the photographs at their highest points is therefore evidence of the disturbing influence of eddy currents. Furthermore, if a method of correcting for eddy currents can be found, the phase relation of the highest points will furnish the criterion for determining the magnitude of the correction. Eddy currents are produced by the variation of the flux, ϕ, and their magnitude could be expressed as $K' (d\phi/dt)$, where K' is a negative constant. The current in the secondary, which operates one loop of the oscillograph, is also proportional to the derivative of the flux with reference to the time and the curves are marked $K (d\phi/dt)$ on the figures. Some fraction of $K (d\phi/dt)$ will equal $K' (d\phi/dt)$, in other words some fraction of the ordinate of the secondary curve on the oscillographic record will constitute the correction that should be applied to the corresponding ordinate of the current curve to give the ordinate that is proportional to the actual field intensity. It is comparatively easy to tell by inspection of the curves what this fraction is; or at least it may be found after two or three trials in applying the correction to the current curve. The flux and its time rate of variation are behind the current in phase at the highest points of the curves. If the fraction chosen be too small, the flux will be behind the corrected curve while if the fraction chosen be too large the flux curve will fall ahead of the new curve in phase at the highest points. A value of the fraction is chosen which places the corrected curve in phase at the highest point with the flux curve and then the ordinates of this new curve represent the resultant or actual field intensity applied to the iron. Fig. 9 shows the corrected curve with points indicated by the small circles. The correction applied to each ordinate of the curve was 13 per cent. of the corresponding ordinate of the secondary curve. The hysteresis for this specimen, as shown by the corrected curves, is found to differ very little from that shown by the other specimens for an equal change of flux.

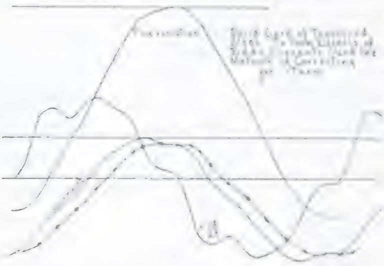

Fig. 9

In order to further test this method of correcting for eddy currents, a solid magnet of tempered steel was used. Eddy currents would certainly be large in this case, but it was arranged to reduce end effects to a negligible magnitude by filling the small air gap with a carefully fitted block of iron. Fig. 10 shows the results. The curve plotted from the corrected data shows no more hysteresis than was found for the wire ring.

To obtain further data on the relative magnitude of the hysteresis in steel and soft iron for the kind of cycle here considered, a ring of soft iron wire was used. Wire having a diameter of .96 mm. was chosen. The wire of specimen *A* had a diameter 1.03 mm. If size of wire were all that determined the eddy currents effect, we should be assured of less disturbance from them with this new ring than was experienced with specimen *A*. However, in getting the total hysteresis curve this is not the case. With soft iron the hysteresis curve is nearly parallel to the axis of ordinates for a long way and hence the time rate of change of flux along this part of the curve is much greater than it is for hard steel where the slope of the curve is much less, hence in plotting the total hysteresis curve of soft iron, we should expect greater errors due to eddy currents than would arise in the case of hard steel. This argument, however, does not apply to demagnetization and recovery curves since the slope does not vary greatly from point to point along the curve and the time required for demagnetization or recovery is 1/120 of a second irrespective of the magnitude of the change. Hence for a given flux variation and for a given size of wire, effects due to eddy currents for soft iron ought not to differ widely from those in steel. Curve *D* of Fig. 12 was obtained from this new ring of soft iron wire. The phase difference at the maximum and minimum points of the current and flux curves was so slight that no attempt was made to correct for eddy currents. Apparently the hysteresis is somewhat larger than for the steel specimens. The conclusion is certainly justified that for cycles of the kind here considered the hysteresis of very hard steel is not greater than that of soft iron for an equal variation of flux density and it may be somewhat less.

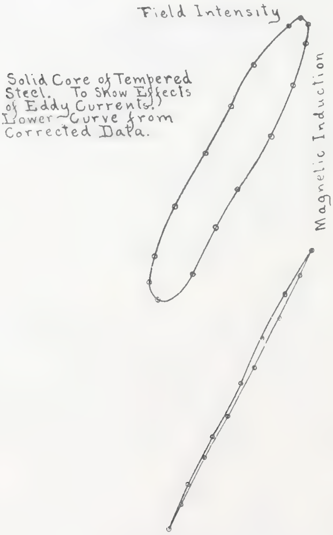

Fig. 10.

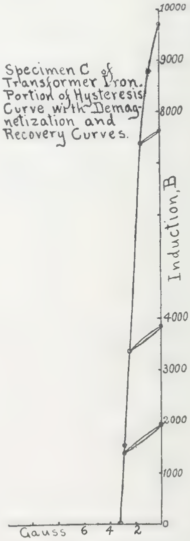

Fig. 11.

The hysteresis curve for ring *C* was plotted ballistically in Fig. 11 and the recovery curves drawn to the same scale as the hysteresis curve are also shown in their relation to the residual magnetic induction. The recovery in the region of small residual magnetism in this case is only about 6 per cent. of the maximum residual induction.

Fig. 12 is for the purpose of comparing the slopes of the curves for the different specimens.

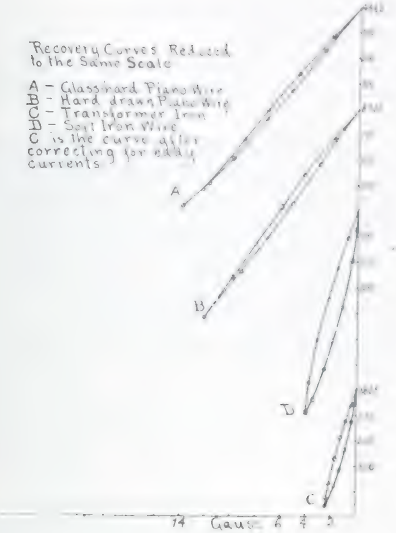

Fig. 12.

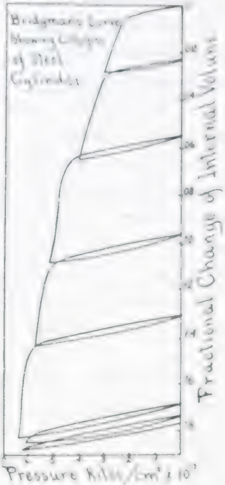

Fig. 13.

C. Conclusions.

1. An oscillographic method has been developed for plotting any kind of a magnetic cycle with an alternating current of 60 cycles provided eddy currents and end effects are negligible. The method is applicable when the flux variation is as small as 150 lines of induction and it may be used for any cycle in which the flux variation is greater than that amount.

2. It was shown that in a given specimen, for any value of residual magnetism, there is a definite elastic limit which is the maximum field that can be applied to the specimen without producing permanent demagnetization; and that if any field not exceeding the limiting field be applied and then removed, the specimen will return to its initial condition.

3. Within the elastic limit, magnetic stress and strain are approximately proportional, *i. e.*, the demagnetization is approximately proportional to the field intensity that produces it, provided the specimen returns to its original remanence when the field is reduced to zero.

4. The hysteresis as indicated by the area of the curves is small and is approximately the same for soft iron and hard steel.

5. The extreme points of the stable demagnetization and recovery curves lie inside the maximum hysteresis curve.

6. The susceptibility to small demagnetizing fields increases slightly with decrease of residual magnetism, the maximum difference observed being about eight per cent.

7. The recovery from the effect of a demagnetizing field is greater for low value of residual induction. For hardened steel it was found to be 1,960 lines of induction per sq. cm. or 24.5 per cent. of the maximum residual induction. For drawn piano wire it was 1,200 lines per sq. cm. or 8 per cent. of the maximum remanence and for soft iron it was 600 lines per sq. cm. or 6 per cent. of the maximum remanence. These numbers represent the elastic limits for low values of residual magnetism.

8. There is shown to be a well-defined upper limit beyond which the residual magnetism of an open magnetic circuit of steel can not be made to pass.

9. The slope of the demagnetization curves for hard steel is 57. For softer specimens it is larger and it reaches a value for very soft iron of about 180.

10. For hard steel the elastic limit is approximately proportional to the difference between the residual magnetism and the maximum residual magnetism.

The work was done in the physical laboratory of the University of Michigan under the direction of Professor K. E. Guthe. The author wishes to express his appreciation of Professor Guthe's interest in the work and his helpful suggestions while it was in progress.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, June 1, 1912.

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ANOMALOUS TEMPERATURE EFFECTS UPON MAGNETIZED STEEL.

BY N. H. WILLIAMS

A SOMEWHAT striking case of anomalous magnetization was described about a year ago by Smith and Guild.[1] Steel rods 7 cm. long were magnetized and then heated. In all cases in which the carbon content was greater than .15 per cent. the magnetism was reversed in the neighborhood of 200° C. The effect was most marked when the percentage of carbon was .85 of one per cent. In that case the negative intensity at 210° was about 10 per cent. of the maximum residual intensity before heating. With further rise of temperature, this negative flux very gradually decreased and finally disappeared at about 700° C.

If, after the temperature had been raised to 210° and a negative magnetic moment established, the specimen were allowed to cool, it partially regained its former positive magnetism.

The explanation of the phenomenon by the authors can be gathered from the following quotation from their paper: "The facts, therefore, suggest the following view of the course of events within each rod. Withdrawal of the magnetizing field leaves the rod subject to the influence of its own self-demagnetizing force, which at first is considerable. This force, acting upon the iron and upon the carbide tends to reverse the polarity of both. The alignment of the iron is more easily disturbed than that of the carbide and a greater amount of reversal is produced in the former. The reversing field diminishes continuously, and when it becomes incapable of producing further change, the feeble residual polarity is made up of a small positive polarity due to the carbide and of a still smaller negative polarity due to the iron.

"When the rod is heated, the polarity of the carbide diminishes more rapidly than the opposite polarity of the iron for several reasons. One of the most important is that the carbide is approaching its transition temperature more rapidly than the iron, and another is that, until the zero moment in the neighborhood of 200° is passed, the demagnetizing field tends to maintain the negatively magnetized constituent."

If this explanation be accepted, we must imagine an intimate mixture

[1] Proceedings of the Physical Society of London, August 15, 1912.

of two different kinds of particles in which the magnetic induction is in one sense in one constituent of the mixture and in the opposite sense in the other; moreover, the reversal of the polarity of one constituent was accomplished by the action of the other.

So long as we think of the substance composing the magnet as a continuous homogeneous medium of definite permeability, it is perfectly correct to say that at an interior point the only field acting is the demagnetizing field due to the ends but when we conceive of the magnet as made up of discrete particles of two kinds of substance in intimate mixture we are no longer entitled to treat the substance of the magnet as a homogeneous, continuous medium. The field brought to bear upon a molecular magnet within the iron would be the field existing in the intermolecular space, which is by no means a demagnetizing field.

Hopkinson's experiments show a large increase of susceptibility of iron and steel to *small* magnetic fields when the temperature is raised. This would lead us to expect considerable self-demagnetization with rise of temperature, but it would not account for the reversal.

The work described in the present paper was undertaken in the hope of furnishing further data upon which to base an explanation of this curious phenomenon.

A solid bar of steel 7 cm. long and having a cross-sectional area of 1.75 sq. cm. was used. The steel contained .85 of one per cent. of carbon. The bar was magnetized by placing it within a helix of many turns of wire, the current being supplied from a storage battery. After the bar was magnetized it was fastened within a frame upon which a test coil was arranged to slide so that it could be brought quickly to a position where it surrounded the magnet. The test coil was connected to a ballistic galvanometer and the frame and coil were immersed in an oil bath. It was assumed that the throw of the galvanometer when the coil was thrown over the magnet or when it was suddenly taken off would be proportional to the residual flux in the metal. The conclusions drawn from the work are not affected if this assumption is considered as only approximately correct. The oil was heated by Bunsen burners and the temperature was measured by a mercury thermometer. In this way the residual flux was measured at brief intervals as the temperature increased from 20° to 220°.

The experiments upon this specimen showed the reversal of polarity that was expected but the magnitude of the reversed flux was not as large in comparison with the maximum residual flux as was obtained by Smith and Guild. Fig. 1 shows the results. Temperatures are plotted as abscissæ and galvanometer deflections (proportional to magnetic flux) as ordinates. The reversed deflection at 195° was about 5 per cent. of that due to the maximum residual flux and upon cooling, the specimen recovered about a quarter of its original positive magnetism. However, the experiment may be so arranged as to avoid the appearance of this

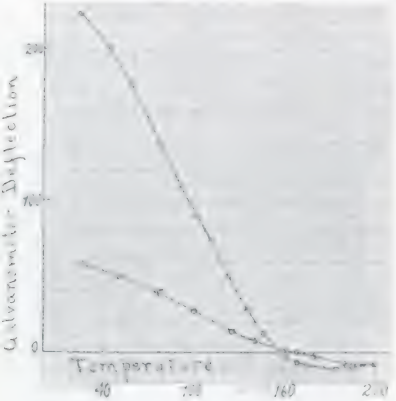

Fig. 2

anomalous behavior. Fig. 2 shows results obtained under the same conditions as those pertaining to Fig. 1 with this exception: This time after magnetizing the steel it was removed from the coil without opening the magnetizing circuit, whereas in the former case the circuit was broken at the switch before the iron was removed from the helix.

It will be observed that in Fig. 2 the flux starts at a very much higher value and diminishes as the temperature rises, coming to zero at about 190°, but that it does not reverse. If the end effects were responsible for the reversal there should have been a larger reversed magnetization in the second case than in the first, since the residual flux which produced the end effect was twice as large in the second case as in the first. Since, however, there was none at all we seem justified in the conclusion that end effects are not the cause of the reversal but that the explanation is to be found in the oscillatory spark at the switch when the circuit is opened. The condition that the discharge shall be oscillatory in a given system is expressed by the relation $R^2/4L^2 < 1/LC$, R being resistance, L inductance and C capacity.

The circuit, here considered, consisted of a storage battery, a coil of wire and a knife switch. The capacity is therefore extremely small and we may be certain that the above condition is fulfilled and that an oscillatory current will flow through the coil when the switch is opened. The period of this oscillation is

$$T = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

or approximately $2\pi\sqrt{LC}$. Since C is extremely small the period is also extremely small.

The effect of such a current on the magnetism of the steel core would be confined to a thin surface layer, the eddy currents in the outside layers being sufficient to protect the inner portion of the core from its action. The first reversed current of this oscillatory discharge would be larger than any subsequent current in either direction and the magnetic effect of this current in conjunction with the demagnetizing field due to the ends might produce a reversal of the outside layers of the core.

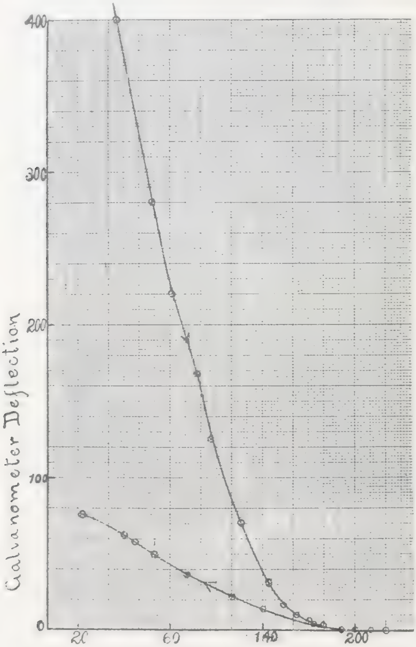

Fig. 2.

It is almost inevitable that steel with such high percentage of carbon should be slightly harder in its outside layers than at the interior of the specimen. We should then expect that with rise of temperature the magnetism of the outer layers would persist after that of the softer interior had disappeared and hence that the resultant magnetic moment would be reversed at the high temperature.

The authors previously mentioned also studied the effect of a rise of temperature upon a specimen that had been first magnetized strongly and then subjected to a demagnetizing field just strong enough to leave the specimen apparently demagnetized. They stated that under these conditions the negative magnetism that developed with rise of temperature was much stronger than in the other cases and that it occurred after a very slight rise of positive magnetism. The author has been unable to reproduce these results. What has happened in each case is a rise of positive magnetism, *i. e.*, a flux in the direction of the original strong magnetizing field and whether the flux finally reverses at 190° is again determined by the method of magnetizing and demagnetizing the specimen.

Fig. 3 shows the results when the specimen was first highly magnetized then demagnetized; in both cases the steel being removed from the coil before the switch was opened thus avoiding entirely the effects of oscillatory currents. With rise of temperature the positive magnetism develops, reaches a maximum at about 80°, then decreases reaching zero at about 200°. On cooling, the steel recovers over 80 per cent. of its maximum flux. There is no indication of a reversal.

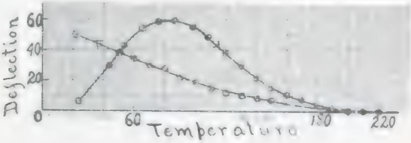

Fig. 3.

Two other cases naturally suggest themselves: (*a*) The specimen might be magnetized and the circuit broken at the switch, then demagnetized without breaking the circuit. We should expect then a reversal of magnetism at high temperatures. (*b*) The specimen might be magnetized and removed from the coil without breaking the circuit, then subjected to a demagnetizing field, the circuit being opened at the switch in the latter operation. The effect of the oscillating current would be to reverse the outer layer with respect to the final field, *i. e.*, to magnetize it in the direction of the original field, which we have called positive. So in this case we should not expect a reversal of magnetism with rise of temperature nor indeed should we expect to be able to reduce the flux to zero in the neighborhood of 200°.

Figs. 4 and 5 represent data obtained under the conditions described under (*a*) and (*b*) respectively, and they are in accordance with the statements there made as to what should be expected.

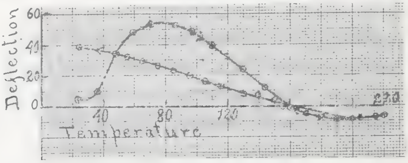

Fig. 4.

A good illustration of the effect of the oscillatory current was found while obtaining data similar to those of the last two curves. The specimen was magnetized strongly in the positive direction and then subjected to a negative field such that if the bar were slowly withdrawn from the field, it retained negative polarity sufficient to produce a throw of the galvanometer equal to − 102. This same negative field was again applied and the current interrupted at the switch, thus giving rise to oscillations. The residual magnetism was then found to be positive and sufficient to produce a deflection + 60. Again the field was applied without permitting oscillations and the residual flux produced a deflection of − 106. Another application of the same field followed by a break at the switch caused positive residual magnetism, and so on indefinitely.

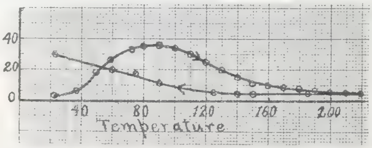

Fig. 5.

If the explanation proposed for this apparent anomaly be correct, the phenomena may be classified with other cases which have been described in detail in a recent paper by B. O. Peirce, on "The Anomalous Magnetization of Iron and Steel."[1]

University of Michigan,
Ann Arbor.

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[1] Proceedings of the American Academy of Arts and Sciences, March, 1912.

POLARIZATION IN THE ALUMINIUM RECTIFIER

POLARIZATION IN THE ALUMINIUM RECTIFIER.

By Clarence W. Greene.

THE primary object of this investigation was to make a study of the potential difference between the electrodes of the aluminium rectifier at extremely short intervals after breaking the charging circuit. It was hoped that the investigation might also throw some further light on the action of electrolytic cells. The methods employed and results obtained are new and may be suggestive in the further study of the electrolytic cell.

The aluminium plates, which were used as electrodes, were prepared in the instrument shops of the University of Michigan and were hammered out of 98 per cent. pure aluminium ingots. They were 10 cm. long, 2.7 cm. wide and 0.3 cm. thick. The plates were so insulated with Bank of England sealing wax as to leave a definite area exposed as active electrode to the electrolyte. The cathode was a platinum plate of 20 sq. cm. exposed area, totally immersed in the electrolyte, the platinum wire leading out of the solution being thoroughly insulated from the solution. The insulation thus eliminated any possibility of surface effects, which are found to be present when the electrode surfaces are exposed at the surface of the electrolyte. The active surfaces of the aluminium electrodes were polished with fine sand paper and rinsed with distilled water before each experiment was begun. The electrolytes used were ammonium aluminium sulphate (40 grams per liter of water) and potassium dichromate (26.18 grams per liter of water).

After checking some of the results obtained by former investigators[1] by the ballistic galvanometer-condenser method, a potentiometer method for measuring the counter E.M.F. was devised and the results thereby obtained were compared with those obtained by the former method. The potentiometer method invariably gave larger values for the counter E.M.F. of the rectifier than did the condenser method, the differences being far beyond the range of experimental error. For instance, with a given E.M.F. the difference was consistently about 4.7 per cent. A repetition of comparative experiments gave a similar characteristic difference, which indicates that when the charged rectifier is connected to a condenser, a quantity of electricity required to charge the condenser being taken from the rectifier, the electrical condition of the rectifier is thereby disturbed and the difference of potential between the electrodes of the rectifier decreased. The potentiometer method was, therefore, used throughout the remainder of the experimental work.

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[1] Theory of the Electrolytic Rectifier, by S. R. Cook, PHYS. REV., 18, pp. 23–29, Jan., 1904; also PHYS. REV., 20, pp. 312–321, May, 1905; H. W. Morse and C. L. B. Shuddemagen, American Acad. Arts and Sciences, *Proc.*, 44, pp. 367–397, 1908–9.

The arrangement of apparatus for the measurement of the counter E.M.F. of the rectifier after the expiration of definite periods of open circuit is indicated in Fig. 1. *R* is the aluminium rectifier, in which *A* is the aluminium anode; *C*, the platinum cathode; and *B*, an aluminium test electrode placed at one side and somewhat in the rear of the anode so as not to be in the path of the current between anode and cathode, *C.S.B.*, the charging storage battery; *M.A.*, a milliammeter; R_1 and R_2, two high grade Leeds and Northrup resistance boxes; R_3, a high grade Leeds and Northrup resistance box with travelling plug; *H.R.*, a 250,000 ohm resistance box; *S.C.* a cadmium standard cell; *G.*, a Leeds and Northrup high sensibility galvanometer having a resistance of 1000

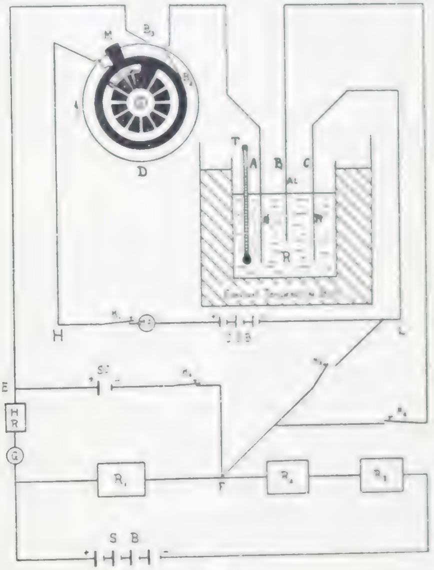

Fig. 1.

ohms; *S.B.*, a storage battery; and *D*, a disc especially designed for this experimental work. The disc consisted of a single piece of hard rubber, 13.5 cm. in diameter and 1 cm. thick, to which were firmly riveted two concentric brass rings, 2 mm. thick and 1.5 cm. wide, the outside diameters being 7.5 cm. and 13 cm., respectively. The inner ring was divided into 30° sectors, one of which was subdivided into two 15° sectors; one of these 15° sectors was later soldered to the adjoining 30° sector, as indicated in the figure. These sectors could be connected, when desired, by inserting a brass plug between them. One of these sectors was electrically connected, as shown in the diagram, to the outer ring. The outer ring was in turn connected to the brass contact maker, *M*, which fitted closely over the rim of the rubber disc. The contact maker could be shifted along the rim by 15° intervals and was firmly held in place by a plug inserted through the contact maker and into the disc from the rear. The disc was rotated by a motor running at an approximately constant speed of 2,000 R.P.M. B_1, B_2, B_3 are spring brass brushes so adjusted as to make good electrical contact with the inner ring, outer ring and contact maker, respectively.

The division of the inner ring into sectors makes it possible to regulate the portion of the period of rotation during which the impressed E.M.F. is applied to the rectifier; the portion of the period finally adopted was the time required for the brush to sweep over a 45° sector. Regulation of the contact maker made it possible to vary the period of decay of the counter E.M.F. of the rectifier before this E.M.F. was balanced against the fall of potential over R_1. The sum of R_1 and R_2 was kept constant and the resistance in R_3 adjusted until the current flowing through R_1 was 0.001 ampere, as was shown, on closing K_s, by balancing the fall of potential over R_1 against the E.M.F. of the standard cell. This made it possible to read the counter E.M.F., on obtaining a balance, directly from the resistance in R_1. On closing K_1, *C.S.B.* charges the rectifier through the disc; on closing K_2 the counter E.M.F. of the rectifier is balanced against the fall of potential over R_1, while B_3 sweeps over *M*.

The arrangement of apparatus indicated in Fig. 1 is such, on closing K_2, as to give the data for the curve of decay between the aluminium anode and the aluminium test electrode, thus giving the effect at the anode only, due to the impressed E.M.F. By closing K_3 instead of K_2 the data for the curve of decay of E.M.F. between anode and cathode were obtained, thus giving the total effect of polarization. To obtain the effect of polarization at the cathode only the positive terminal of *C.S.B.* was disconnected from the disc and connected directly to the anode, while the negative terminal of *C.S.B.* was connected to the cathode through the disc.

It was ascertained that in order to obtain the maximum counter E.M.F. of the rectifier by charging it through the disc while revolving, it was necessary that the motor run for a period of from 10 to 15 minutes before the polarization was measured and 15 minutes was the interval accordingly adopted before any measurements were taken. One of the greatest difficulties met with in experimental work with the rectifier is the slow change in the character of the film with time; and it is therefore necessary (1) that the series of results be taken as rapidly as possible, (2) that the period of open circuit between the determinations of the balance for successive intervals of decay be kept constant, (3) that the points on the decay curve be determined in the same order. A cyclic order of performing the operations was followed, thus rendering it feasible for each series of results obtained to be checked. Yet the author believes that the curves obtained are characteristic of the general behavior of the anode film.

Table I. gives the mean corrected values obtained from a number of experiments and Fig. 2 gives the curves plotted therefrom for the decay of the counter E.M.F. between the anode and the aluminium test electrode ($A-3$), between the anode and cathode (AC) and between the cathode and aluminium test electrode ($C-3$), for an applied E.M.F. of 6.16 volts. Table II. gives the corresponding data and Fig. 3 gives the curves for an applied E.M.F. of 12.25 volts. Table III. and Fig. 4 give the corresponding results for an applied E.M.F. of 18.13 volts.

TABLE I.

Electrolyte: Ammonium Aluminium Sulphate.
Applied E.M.F.: 6.16 volts.
Temperature: 23° C.
Current: 0.00025 ampere

Period of Decay, Second.	E_{A-3}, Volts.	E_{AC}, Volts.	E_{C-3}, Volts.	$E_{AC}+E_{C-3}$, Volts.	$E_{A-3}-(E_{AC}+E_{C-3})$, Volts.
0.0000	6.43	6.15			
0.0013	6.20	5.90	0.31	6.21	−0.01
0.0025	6.02	5.66	0.36	6.02	0.00
0.0038	5.91	5.50	0.39	5.89	+0.02
0.0050	5.78	5.36	0.35	5.71	0.07
0.0075	5.57	5.17	0.36	5.53	0.04
0.0100	5.43	5.01	0.35	5.36	0.07
0.0150	5.24	4.80	0.38	5.18	0.06
0.0200	5.08	4.64	0.38	5.02	0.06
0.0250	4.90	4.51	0.40	4.91	−0.01

TABLE II.

Electrolyte: Ammonium Aluminium Sulphate.
Applied E.M.F.: 12.25 volts.
Temperature: 23.°2 C.
Current: 0.00035 ampere.

Period of Decay, Second.	E_{A-3}, Volts.	E_{AC}, Volts.	E_{C-3}, Volts.	$E_{AC}+E_{C-3}$, Volts.	$E_{A-3}-(E_{AC}+E_{C-3})$, Volts.
0.0000	12.52	12.31	....	.....	....
0.00083	11.92	11.72	0.31	12.03	—0.11
0.0021	11.32	11.07	0.31	11.38	—0.06
0.0033	11.00	10.69	0.31	11.00	0.00
0.0046	10.69	10.42	0.32	10.74	—0.05
0.0071	10.37	10.02	0.33	10.35	+0.02
0.0096	10.14	9.71	0.35	10.06	0.08
0.0146	9.60	9.28	0.36	9.64	—0.04
0.0196	9.37	8.89	0.37	9.26	+0.11
0.0246	9.23	8.75	0.36	9.11	0.12

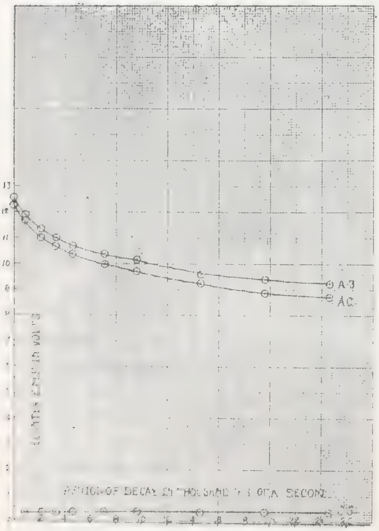

Fig. 2.

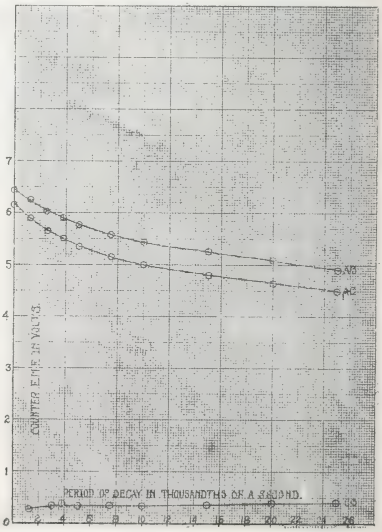

Fig. 3.

It is worthy of note that the polarization E.M.F. between the anode and the aluminium test electrode exceeds the applied E.M.F. by about 0.3 of a volt. The excess voltage is due to the fact that the electrolytic cell, acting as a primary cell, furnished an effective voltage of about 0.3

TABLE III.

Electrolyte: Ammonium Aluminum Sulphate.
Applied E.M.F.: 18.13 volts.
Temperature: 24° C.
Current: 0.0005 ampere.

Period of Decay, Second.	E_{A-3}, Volts.	E_{AC}, Volts	E_{C-3}, Volts.	$E_{AC}+E_{C-3}$, Volts.	$E_{A-3}-(E_{AC}+E_{C-3})$, Volts.
0.0000	18.13	18.07			
0.0013	17.01	16.59	0.34	16.93	+0.08
0.0025	16.11	15.68	0.32	16.00	0.11
0.0038	15.57	15.12	0.31	15.43	0.14
0.0050	15.29	14.70	0.31	15.01	0.28
0.0075	14.63	14.09	0.32	14.41	0.22
0.0100	14.00	13.67	0.31	13.98	0.02
0.0150	13.33	12.91	0.34	13.24	0.09
0.0200	12.98	12.53	0.36	12.89	0.09
0.0250	12.75	12.34	0.37	12.71	0.04

volt in series with the charging storage battery to assist in the process of polarization. The difference between E_{A3} and $E_{AC}+E_{C3}$ for corresponding decay intervals is also seen to be negligible, the error rarely exceeding 1 per cent., which demonstrates that the method used renders possible the attainment of a high degree of accuracy. The results show conclusively that the E.M.F. of polarization, at the instant when the charging circuit is opened, is equal to the polarizing E.M.F., and that there has been no considerable IR component in the surface film during the charging period.

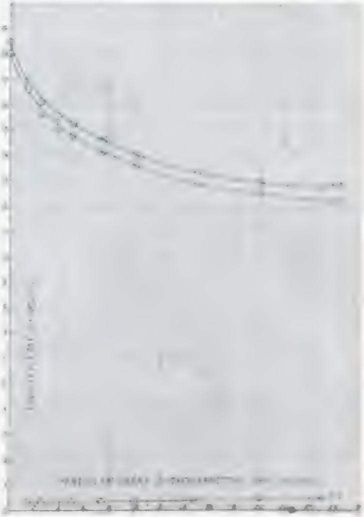

Fig. 4.

It has been proved by Schulze,[1] in an extensive series of experiments, that the effective anodic layer, in conformity with the suggestion of

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[1] Ann. d. Physik, (4) 9, p. [illegible] 1907.

Guthe,[1] consists of oxygen gas. An extension of the gas film theory accounts for the slight residual current that flows through the electrolytic cell. The negative ions accumulate in the electrolyte next to the gas layer and electrons are forced from some of the carriers by the high potential gradient. These free electrons pass through the gas layer, thus causing the observed small residual current. The higher the value of the impressed voltage, under the experimental conditions maintained by the writer, the higher is the potential gradient through the gas film, and the larger should be the number of detached electrons and the resulting current. The results are seen to conform to this theory, since, for example, the residual current for an applied voltage of 6.16 volts was 0.00025 ampere; for 12.25 volts, 0.00035 ampere; for 18.13 volts, 0.00050 ampere.

The above way of looking at the counter E.M.F. of the aluminium cell very closely corresponds to our ideas of the process of charging a condenser and by many experimenters the aluminium anode is spoken of as a condenser. This comparison should, however, not be carried too far. When the two plates of a charged condenser are connected by a conductor of very high resistance, the potential difference between the plates decreases in accord with the well-known law: $V = V_0 e^{-t/RC}$. Passing to logarithms, Log V = Log $V_0 - t/RC$. If the action of the rectifier were that of an ordinary condenser, a straight line should be obtained by plotting the logarithms of V as ordinates and the corresponding periods as abscissæ. Applying this test to the data of Table IV., for instance, curve C, of Fig. 5, is obtained. It shows that the rate of decrease of counter E.M.F. is not such as to indicate that the rectifier conforms to the law of the ordinary condenser. The explana-

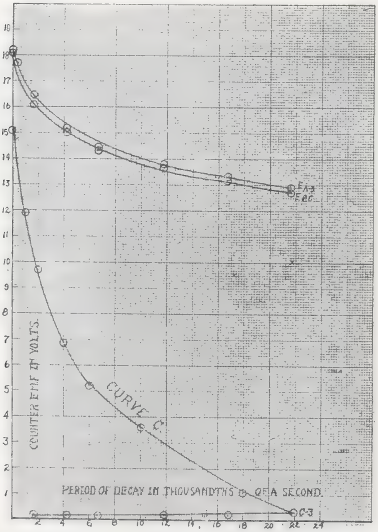

Fig. 5.

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[1] Theory of Electrolytic Rectifier, PHYS. REV., 15, pp. 327–334, 1902.

tion of this discrepancy may lie in the fact that the decrease of potential in the rectifier is due to the diffusion of the ions back into the solution rather than to ordinary metallic conduction.

TABLE IV.

[illegible]
Temperature: 23.1° C.
Applied E.M.F.: 18.16 volts.
Current: 0.0005 ampere.

Period of Decay. Second.	[illegible] Volts.
0.0000	18.22
0.0004	17.70
0.0017	16.31
0.0042	15.17
0.0067	14.44

When the impressed voltage causing the polarization of the rectifier is increased the ionic concentration near the film is increased and with it the rate of change in the concentration passing from the film to the body of the electrolyte is increased. Therefore, when the impressed voltage is removed, the ions should disappear from the boundary of the gas layer the more rapidly the higher the impressed voltage. A comparison of the curves of Fig. 6 confirms this view. If the assumption that the rate of diffusion of the ions from the gas layer back into the solution is a function of the difference between the ionic concentration near the anode and that in the remainder of the electrolyte, is correct, — then as the process of diffusion proceeds and the difference in ionic concentration decreases, the rate of diffusion (or the rate of decay of the counter E.M.F. of polarization) should decrease. This is seen to be true of each of the curves of decay.

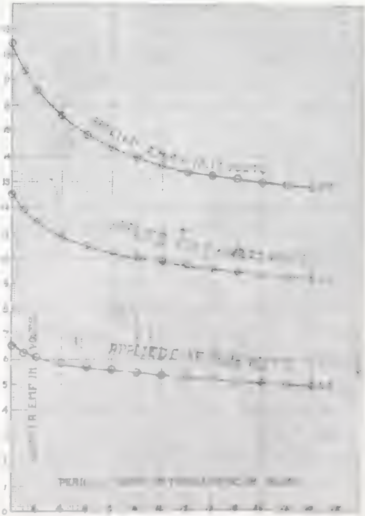

Fig. 6.

It is believed that the curves of decay obtained represent with a fair degree of accuracy no other effects than those of ionic diffusion, since during the extremely short intervals of decay used the effect of any disintegration of the film may be assumed to be negligible.

The results of this investigation may be summarized as follows:

1. The condenser method of measuring the counter E.M.F. introduces an error due to the taking of a charge from the rectifier.

2. The rectifier does not behave as an ordinary condenser.

3. When aluminium is used as the anode, the counter E.M.F. alone accounts for the reduction of the current to its exceedingly small value.

4. The results obtained in this investigation are apparently in accord with the gas film theory and the theory of ionic diffusion.

The writer is indebted to the University of Michigan for the special apparatus provided for the work and to Professor K. E. Guthe, of the University of Michigan, for many helpful suggestions received.

Physical Laboratory,
University of Michigan.

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A STUDY OF THE LONGITUDINAL VIBRATION OF WIRES.

By George A. Lindsay.

I. Introduction.

THE elastic behavior of wires has been very carefully investigated by means of torsional vibrations. This method lends itself most readily, perhaps, to the study of internal friction, for the period may easily be regulated to any desired length, the amplitude may be made large, and, by suitable adjustment of the suspended mass, the free vibration may be continued for a comparatively long time.

Weber[1] was the first to observe that a vibrating body is damped in a vacuum, thus showing with certainty the existence of an internal cause of damping, and O. E. Meyer[2] made the first quantitative study of the phenomenon, by observing the damping of torsional vibrations.

Voight[3] measured the damping of flexural, as well as torsional vibrations, with the view of determining whether, according to Boltzmann's theory, the logarithmic decrement is independent of the period, as the latter shows it should be if damping is caused by the elastic after-effect; or whether there is really internal friction, in which case the logarithmic decrement should depend on the period. The results of his work were not entirely in favor of either theory. For flexular vibrations he found that, for copper and some other metals, the logarithmic decrement varied inversely as the period, thus supporting the internal friction theory, while with aluminum, cast iron, and cadmium the logarithmic decrement was much more nearly constant. In a few cases the decrement actually increased with increasing period, which is at variance with both theories. His conclusion was that in some cases the preponderant factor in damping is internal friction, in other cases it is the after-effect. Voight also found, as Schmidt[4] had done earlier, that the logarithmic decrement varies approximately as the square of the amplitude. Streintz,[1] however, found it to be independent of the amplitude in torsional vibrations, and also of the period, if the period were varied by changing the moment of inertia of the suspended mass. Guthe and Sieg[2] found certain platinum-iridium wires, when vibrated torsionally, to give a maximum value of the logarithmic decrement, and the position of the maximum on the decrement-amplitude curve depended on the initial amplitude. This work will be referred to more in detail later in comparing with the damping of longitudinal vibrations. Harris,[3] using bismuth wires, observed a decrease of the logarithmic decrement with decreasing amplitude, but no maximum or minimum values.

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[1] Pogg. Ann., 34, p. 247, 1835.

[2] Ibid., 113, pp. 77, 193, 1861.

[3] Abh. der Königl. Gesell. der Wiss. zu Göttingen, 38, 1892.

[4] Wied. Ann., 2, p. 48, 1877.

Other experiments, notably those of Streintz, Wiechert,[4] and Bouasse[5] and Carriere, have shown that the previous history of the wire very markedly influences the after-effect and the damping. While the after-effect has been carefully observed for longitudinal deformation, as far as the writer has been able to find, little or nothing has been done on the problem of the quantitative measurement of the damping of longitudinal vibrations.

The present investigation is an attempt to measure as accurately as possible the period of longitudinal vibrations of wires, to consider possible causes of difference between this and the theoretical period, and to measure the damping of the vibrations. Both static and dynamic observations will be given which indicate that the modulus of the wires is not exactly a constant with varying extension. An expression for the period involving the amplitude will be derived, which shows that this variation of the modulus cannot account for the observed variation of the period with amplitude. It will also be shown that, with the ordinary assumptions, the variation of the logarithmic decrement with amplitude is not sufficient to explain the variation of the period.

II. ELONGATION OF THE WIRES.

Apparatus.

Four wires were used; one each of copper, steel, phosphor-bronze, and platinum-iridium. The steel wire was new piano-wire; the copper, a piece from a spool of commercial wire. Three of the four wires were about 230 cm. long, while the fourth, the platinum-iridium, was only about

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[1] Wien. Ber., 69, II. Abt., p. 337, 1874: 80, II. Abt., p. 397, 1879.

[2] PHYS. REV., Vol. 30, No. 4, 1910.

[3] PHYS. REV., Vol. 35, No. 2, 1912.

[4] Wied. Ann., 50, p. 546, 1893.

[5] Ann. de Fac. des Sci. de Toulouse, II., 431; III., 217; IV., 357. Ann. de Chimie et de Physique, 8me Serie, t. 14, p. 190.

140 cm. long. No attempt was made to take account of previous treatment of the wires, since the adjustment for vibration was rather difficult, and it was usually necessary to vibrate the wire for some time before measurements could be made. Excepting the platinum-iridium, which contained 40 per cent. of iridium, and was prepared by Dr. Heraeus, of Hanau, the composition of the wires was not well known, and it is not to be expected that the results are rigidly characteristic of these metals. In fact, it is known that hardness has a great influence on the internal friction of wires. Annealed wires exhibit greater damping than tempered ones of the same material.

A heavy iron bracket was fixed rigidly to the brick wall of the laboratory. A strap of iron was bolted firmly to the front of the bracket; but this was not used as a clamp, for clamping the wire would flatten it, and weaken it at the point where it entered the clamp, besides rendering uncertain the location of the upper end of the vibrating length. Instead, a small slot, only large enough to admit the wire easily, was sawed in the iron strap. This slot was filled with solder, and the upper end of the wire was soldered in, thus attaching the wire very firmly to the support. In order to keep the temperature uniform and fairly constant, the wire was enclosed in a wooden box, the internal cross-section of which was about 16 cm.^{2} The upper end of the box was lightly covered with cotton batting to prevent air currents passing through, yet not so as to interfere with the vibrations. Two thermometers were inserted through the side of the box, one near the top and the other near the bottom.

A fan was installed near the box to keep the air in the room in circulation. Without the fan the temperature of the upper part of the box sometimes became as much as 3 degrees higher than the lower end, especially in colder weather, when the room was heated by pipes near the ceiling. When the fan was working the upper and lower parts of the box differed usually by not more than 0°.1 C. A variation of one or two degrees has very little effect on the period of vibration, since the modulus changes only slightly with the temperature; but in determining statically the elongation due to a certain load, it was very important that the temperature should not change between readings.

The following precautions were taken to straighten the wires before using. The steel and phosphor-bronze wires were suspended and stretched for about one day by ½ the maximum load which they were required to carry. The copper wire was very soft, and was straightened by passing it through a drawplate, using a hole the size of the wire. This straightened the wire without drawing it any smaller. The platinum-iridium wire was moderately loaded and annealed by passing through it an electric current sufficient to raise it to a yellow heat.

Instantaneous Recovery.

In order to make a comparison of static and dynamic moduli, the simple method of measuring the elongation directly was chosen. Consider a wire loaded as in Fig. 1. Let P be the position of the lowest point of the wire when at rest. From the elastic forces called out by the distortion, P has a tendency to return to some point O. This point may not be at all the position of P before the load was imposed. It does not even remain constant when P moves up and down. The variation of O is due to two things: (1) the elastic after-effect; (2) the heating effect of varying the length of the wire. When the wire is at rest, (2) is not to be considered; therefore, disregarding this for the present, let us call $OP = e$ the instantaneous recovery. This is evidently the distance on which the restoring force depends, and this is the distance it is desired to measure.

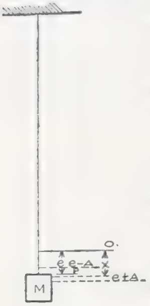

Fig. 1.

Lever Systems for Measuring Recovery.

In order to determine changes occurring very soon after unloading, the device shown in Figs. 2, 3, and 4 was used. Figs. 2 and 3 show two sections at right angles to each other. Fig. 4 shows one of two similar levers, made each of two stips of brass bent and soldered together along one half their length. Small conical hollows were made at *ee*, Fig. 4, to receive the points of p in Fig. 3. The arms of the lever were bent just far enough apart so they would hold on the axle p without play, and with little friction. The lever was thus free to move about the axis of p in a vertical circle. L, Fig. 2, is this lever seen from the side. d is a small closely fitting pin to one end of which the wire W was lightly soldered. m is a small mirror with its plane at right angles to the lever, and containing the axis of p. One of the levers was used for the wire under test, the other for a comparison wire hung about 2 cm. from the first. The mirrors on the two levers were placed so close together that by means of a single telescope two scales could be viewed at the same time, one reflected from each mirror.

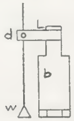

Fig. 2.

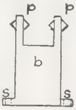

Fig. 3.

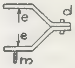

Fig. 4.

Wire No. 2 indicated any sagging of the bracket, and change of temperature. It was found that with the fan running and the box in position the temperature change during the readings necessary for a single determination of the elongation was ordinarily too small to be observed. For measuring the elongation the mass of figure 1 was replaced by a scale pan.

A mass was placed on the pan and after some time a reading was taken on both scales, and the mass was immediately removed. Then readings were taken on the wire under test at the end of 5 sec., 15 sec., 1 min., 2 min., and 3 min. Two or three readings on the comparison wire gave corrections to be applied for sagging of the support, and for temperature changes. The comparative rigidity of the lever system allowed a reliable reading to be taken 5 sec. after removal of the mass. The results are shown for the copper wire in figure 5. The numbers in parentheses refer to different masses removed from the pan as shown in the accompanying table.

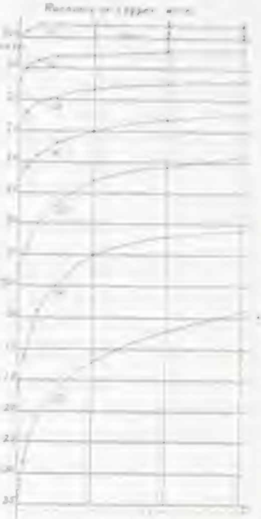

Fig. 5

Copper.

	Mass, gms.
(1)	471
(2)	940
(3)	1,401
(4)	1,872
(5)	2,343
(6)	2,815
(7)	3,295

Each mass was left on the pan 2 min. The relative positions of the curves in Fig. 5 show how slowly the copper wire recovered from strain. The after-effect and the heating effect are both present in these curves. The curve for a time interval less than 5 sec. is unknown, but producing it as the observed part seems to indicate we would have an intersection on the scale axis as shown by the dotted line.

Owing to the various quantities upon which the computed elongation depends when deduced from the scale readings and the possibility of error, especially in the motion of the pin *d*, shown in Fig. 2, it was thought best to depend on the above apparatus only for the small changes taking place in the first minute or two, and to measure the entire recovery of the wire simply by means of a micrometer microscope focused directly on a point of the wire. Then the correction shown by Fig. 5, and that for the sagging of the bracket were applied afterward.

For all except copper it was seen that in one half min. to one min. the wire had entirely recovered from the strain. The microscopic readings for these could therefore be taken soon after removal of the load. With the copper they were all taken long enough after release so that the change in length had nearly ceased, but the time of each reading was noted, and the proper correction applied from the plots.

Instantaneous Recovery Independent of Period of Deformation.

It was found by means of the lever system that the length of time the load remained on the pan did not influence the instantaneous recovery. The following table shows the instantaneous recovery for various periods of deformation of the copper wire. t is the length of time the weight remained on the pan, e the instantaneous recovery in cm. of the scale. It is the difference between the intersection of the curve with the scale axis (see Fig. 5) and the reading just before release.

TABLE I.

Load = 2 Kg.

t min.	e.
1	11.25
5	11.23
15	11.20
30	11.22
60	11.22
30	11.23
15	11.21
5	11.20
1	11.23

This indicates that although the wire may suffer a gradual change when distorted by a constant force, a given point of the wire has always the same instantaneous recovery.

Since the change observed by aid of the levers and mirrors after the removal of the load is small compared with the total recovery, we may compute the change s in the length of the wire, corresponding to the distance OD shown on the plot of Fig. 5, by the equation

$$s = \frac{1}{2}\frac{r}{a} OD, \tag{1}$$

where r is the radius of the lever arm and a is the distance from mirror to scale. r was determined by measuring, with a microscope, the diameter of the circle in which the pin *d* moves when the lever is rotated about the axis. *r* was thus found to be 2.124 cm.

Temperature Change Due to Loading and Unloading.

The formula for temperature change given by Sir William Thomson[1] is

$$t = \frac{AT\alpha}{\omega c}\Delta p, \tag{2}$$

where A = the reciprocal of Joule's equivalent.
T = abs. temp.
α = coeff. of linear expansion.
ω = linear density of the wire.
c = specific heat.

Edlund[2] found, experimentally, that this formula gave too high a value for t, and according to his results $A = \frac{1}{68270}$ when Δp is in gm.

When the load is suddenly removed the wire is heated, and as it cools it contracts, which contraction added to the after-effect gives the curves in Fig. 5. To compute the part due to the cooling of the wire we have the change of length due to a change in temperature t,

$$\Delta l = \frac{AT\alpha^2 L}{\omega c}\Delta p. \tag{3}$$

If we take $L = 200$ cm.,

and $\Delta p = 3{,}000$ gm., Table II. shows the Δl of equation (3) for each of the four wires. Edlund's value of A is used.

TABLE II.

Wire.	Δl cm.
Steel	0.0010
Copper	0.0004
Phos. br.	0.0020
Pt.-ir.	0.0008

The instantaneous recovery of the wires is given by

$$e = e' - s + \Delta l, \tag{4}$$

where e' is the elongation as measured by the microscope directly. Δl must be determined for the particular length used in obtaining e, then if this is not the same as the length used in measuring e', $e - \Delta l$ must be reduced to that length before subtracting from e'. For the phosphor-bronze and platinum-iridium $s = \Delta l$ as nearly as the plots will show. For steel, s was a little more than twice Δl, and for copper, nearly ten times Δl. This indicates that phosphor-bronze and platinum-iridium have no after-effect, yet this supposition does not agree with the results of timing. Reasons will be given later for supposing that all the wires have an elastic after-effect.

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[1] Math. and Phys. Papers, Vol. 3, p. 66.

[2] Pogg. Ann., 126, p. 539. 1865.

Variation of Instantaneous Recovery with Temperature

To find how the instantaneous recovery varies with the temperature, which is equivalent to finding the variation of the modulus with temperature, the room was kept successively at ten different temperatures ranging from 15°.7 to 26°.4 C., and the instantaneous recovery was determined for each temperature for all the masses concerned in Fig. 5. This test was made for copper only, for the variation of the modulus with temperature is much smaller for the others. According to Pisati it is only one part in 10,000 per degree for steel at temperatures from 0° to 50°. With copper, however, it is large enough to be detected over the range of only 10° employed here, and by plotting instantaneous recoveries and temperatures the equation

$$e_t = e_{23}[1 + 0.0009\,(t - 23)] \qquad (5)$$

was found for the relation between the recovery and the temperature.

Since the word elongation is not to be used in any other sense in this paper, "elongation" will henceforth be used to denote the instantaneous recovery.

Rapidity with Which the Wire Regains Normal Temperature.

The heat evolved by the removal of the load is soon dissipated. A test was made on the copper wire, which, with a diameter of 0.5 mm., was much the largest wire of the four, to see how much time was required for the wire to regain its original temperature. An iron-constantan thermocouple was connected to the wire by separating one of the junctions, passing the separated elements through small holes in the side of the box, and soldering them as lightly as possible to the copper wire, a few cm. apart. The junctions were covered with a little cotton batting so they might cool at most no faster than the remainder of the wire. The other junction of the thermocouple was kept at a constant temperature outside the box. The cross-section of the thermocouple wire was about 1/30 that of the copper wire to which it was attached.

When a mass of 2 kg. was removed from the pan, the deflection of a galvanometer in the thermocouple circuit had practically all disappeared in one minute, showing that in that time the wire had regained its normal temperature. Since the other wires were of less than one half the diameter of the copper wire, the rise in temperature would no doubt disappear in a much shorter time for them.

Results for Elongation.

In Table III. are given the elongations for the various wires. The values of e for copper were all reduced to 23° by formula (5). While the entire set of elongations for copper was taken at 21°, the wires were vibrated at higher temperatures, and it was for comparison with the vibration results that the reduction was made. For all the other wires the variation of modulus with temperature was neglected.

M and e are corresponding masses and elongations, the zero for each being taken with the pan alone suspended.

M includes one half the mass of the wire.

Mass of the pan = 223 gm.

L' = length for which the elongation was measured.

L = vibration length.

e = elongation for the length L.

D = diameter of the wire.

Table III.

Copper.
$L' = 209.7$ cm.
$L = 232.1$ cm.
$D = 0.586$ mm.
Temp. 21°.0

M Gm.	e Cm.	e_{23} Cm.	M/e.
957	0.0821	0.0822	11,640
1,387	0.1182	0.1184	11,710
1,818	0.1560	0.1563	11,630
2,248	0.1941	0.1945	11,560
2,679	0.2315	0.2320	11,550
3,101	0.2690	0.2695	11,510

Steel.
$L' = 154.1$ cm.
$L = 231.1$ cm.
$D = 0.217$ mm.
Temp. 24°.2

M Gm.	e Cm.	M/e.
1,000	0.2874	3,478
1,500	0.4309	3,480
2,000	0.5766	3,467
2,500	0.7213	3,465
3,000	0.8678	3,457

Phosphor-bronze.
$L' = 187.6$ cm. $D = 0.239$ mm.
$L = 231.5$ cm. Temp. = 22°.5.

M Gm.	e Cm.	M/e.
953	0.4138	2,303
1,383	0.6023	2,296
1,814	0.7912	2,293
2,244	0.9816	2,286
2,675	1.1742	2,278

Platinum-iridium.
$L' = 140.5$ cm. $D = 0.200$ mm.
$L = 144.6$ cm. Temp. = 22°.5.

M Gm.	e Cm.	M/e.
953	0.1461	6,523
1,383	0.2119	6,527
1,814	0.2780	6,525
2,244	0.3448	6,508
2,675	0.4137	6,466
3,097	0.4812	6,436

M/e has no significance for the material of the wire, since it contains the length and diameter, but as long as these are constant it serves to indicate whether Young's modulus is constant or not. Owing to the shortness of the wires used, no great degree of absolute accuracy is claimed for the above elongations, and also, since the computation of the period of vibration does not demand it, Young's modulus has not been computed. Its absolute value is of little importance here, but its variation is of great importance.

III. DETERMINATION OF FREQUENCY OF THE VIBRATIONS.

Apparatus.

The pan was removed from the wire, and in its place was soldered a brass disk 6½ cm. in diameter, provided with threads on the circumference, so that over it might be screwed a hollow cylindrical shell about 14 cm. long. Inside this shell could be attached five other solid disks, making six different masses which could be used in vibrations. The shell was closed at the bottom, so that for all masses used, the same surface was exposed to the air and the damping due to the air friction was, therefore, constant.

From the lower end of the cylinder a brass rod 8 mm. in diameter projected vertically downward an additional 10 cm., and carried on its lower end a small soft iron armature. The sections of the mass separated from the cylinder are shown in Fig. 6. The electric circuit was arranged, as in Fig. 7, so that the wire was self-driven after the manner of the electrically driven tuning fork. The switch *L* closed both circuits at once. The vibration could be started by raising or lowering the adjustable cup *g* until, when the proper height was reached, the vibration would start and the amplitude could be controlled by the rheostat.

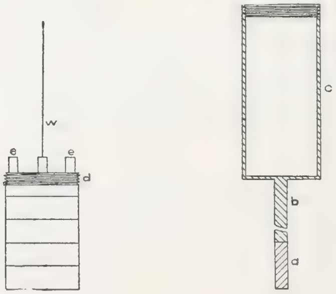

Fig. 6.

a, soft iron armature; *c*, cylindrical shell; *d*, upper disk which screws into the shell.

A tuning fork supplied with adjustable masses for regulating the period, and carrying a lens on one prong, was arranged to vibrate horizontally in front of the wire, so that when a drop of mercury was placed on the wire a Lissajous' figure could be viewed through the vibrating lens by means of a microscope. A phonic wheel was included in the circuit with the tuning fork, so that the fork could be accurately rated while the wire was vibrating. The vibration was so slow that the time required for an elastic pulse to travel the length of the wire was very short compared with the period of the wire, therefore we may consider the wire to be uniformly strained throughout its entire length at all times.

Measuring the Free Period.

This, of course, was a forced vibration of the wire, and the period was therefore slightly less than that of the free vibration. It was found, however, that by adjusting so that the fork was slightly too slow, then

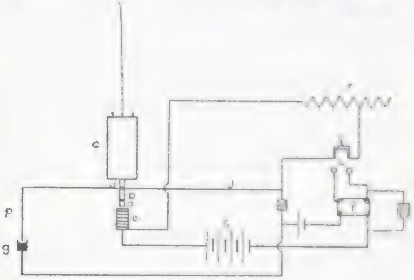

Fig. 7.

e, electromagnet actuated by battery *s*; *f*, relay; *g*, mercury cup; *j*, fine wire soldered to moving system at *d*; *p*, stiff platinum-tipped wire for contact with mercury at *g*.

shutting off the current driving the wire, the free period could be obtained with considerable accuracy.

The free vibration could be observed easily for 1¼ min. with the copper wire, about 4 min. with the platinum-iridium wire, and 5 min. with the steel and phosphor-bronze. In case the period varied with the amplitude, as it did with the copper and platinum-iridium, and also to a slight extent with the phosphor-bronze, the fork was set too slow for small amplitudes, and too fast for large ones. By means of the micrometer microscope the amplitude for which the Lissajous' figure was at rest could be satisfactorily determined. This, however, necessitated changing the rate of the fork, and so rating again by the phonic wheel for every amplitude measured, an expenditure of time which was partly avoided by counting the number of seconds elapsing during one cycle of the Lissajous' figure and taking for the mean amplitude, the average of the amplitudes at the beginning and the end of the count.

Relation of Frequency to Amplitude.

For the steel wire no variation of the frequency with amplitude could be detected. The phosphor-bronze showed a very slight increase in frequency with decreasing amplitude, the variation for platinum-iridium was more marked, and the copper wire showed the greatest variation of all. In all cases the plot of frequency and amplitude was practically a straight line, which, produced to the frequency axis, gave the frequency for infinitely small vibrations. The equation connecting frequency and amplitude is $n = n_0 - bA$, where b is a positive number. The relation is shown graphically for part of the observations in Figs. 8 and 9. The numbers (4), (5), (6), in parentheses, refer respectively to the largest three masses given in Tables V. and VI., in order of increasing magnitude. The cause of this variation of period with amplitude will be discussed later.

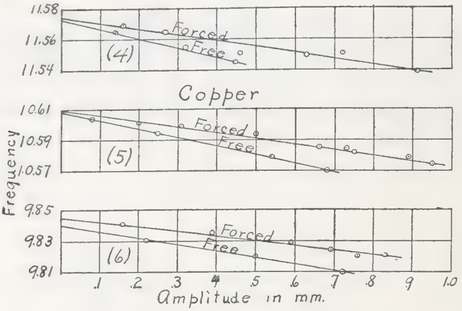

Fig. 8.

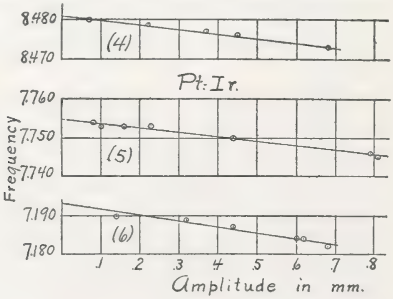

Fig. 9.

Free and Forced Vibrations.

In Fig. 8 for copper are included the plots of the forced frequency. It is seen that this also gives a straight line of nearly the same slope as that of the free vibrations, and the forced period differs very little from the free. That there is not a greater difference is due to the fact that the relay included in the circuit aided the induction in delaying the pulse given by the electromagnet. If the pulse were given just at the instant when the armature reached the middle of its downward motion, there would be no difference of the free and forced periods.

Although the force acting on the armature is not simply harmonic with respect to the time yet it may be represented by a Fourier's series, and it may be shown from the difference between the forced and the free periods that for small amplitudes the fundamental of the Fourier's series is in the neighborhood of 65° behind the motion of the armature.

Table IV. is given as an illustration of the manner of timing.

Table IV.

Copper.

$M = 2{,}678$ gm. $\rho = 5/1$. Temp. = 21°.9 C.

N = frequency of fork.
n = frequency of wire.
$\rho = N/n$.

N	Forced.		Free.		Sec. per Cycle.	Wire Slow or Fast.
	Amp. (Mm.)	n	Amp. (Mm.)	n		
			0.08	10.604		
53.020	0.20	10.602			102	slow
	0.31	10.598			34	slow
			0.25	10.593		
	0.50	10.593			∞	—
52.964	0.66	10.586			31	slow
	0.73	10.583			21	slow
			0.44	10.578		
	0.75	10.582			44	fast
52.888	0.89	10.578			∞	—
	0.95	10.574			50	slow
52.860			0.60	10.573		

The 6th column shows the number of seconds required for the Lissajous' figure to pass through one cycle, while the last indicates whether the wire was faster or slower than 1/5 the rate of the fork.

Tables V. and VI. show in condensed form the free amplitudes and frequencies for the various masses.

Such tables are not given for phosphor-bronze and steel for reasons already stated.

TABLE V.

Copper.

Mass, Gm.	Amp., Mm.	n	p	Temp., C.
956	0.10	17.700	5/2	23°.7
	.15	17.693		
	.33	17.676		
1,386	.17	14.715	3/1	23°.8
	.32	14.696		
	.59	14.672		
1,817	.19	12.853	7/2	24°.0
	.24	12.848		
	.40	12.839		
2,247	.14	11.565	4/1	24°.0
	.32	11.555		
	.45	11.545		
2,678	.08	10.604	5/1	21°.7
	.25	10.593		
	.54	10.578		
	.68	10.572		
3,100	.22	9.831	5/1	23°.8
	.50	9.818		
	.72	9.810		

Variation of Frequency with Mass.

Since, if Hooke's Law holds the period is proportional to the square root of the mass, we should have the product of mass and the square of the frequency equal to a constant. From Table III., the ratio M/e is not a constant. It decreases with increasing mass, except for the first two masses of the three wires, copper, steel and platinum-iridium. It would not be concluded from this alone that the elongation is not proportional to the weight removed, for there is considerable uncertainty about the point to which the curves of Fig. 5 should be produced before they reach the axis of ordinates. However the results of timing also show that M/e is not a constant, for Mn^2 is not constant, except for the case of copper. Table VII. shows the masses, frequencies, and Mn^2, for each of the four wires.

The n for the copper and the platinum-iridium is obtained from Figs. 8 and 9. It is the n indicated by the straight line, for the amplitude zero. For copper a small correction to n was made to reduce it to 23° temperature, the same temperature as that for the elongations of Table

TABLE VI.

Platinum-iridium.

Mass, Gm.	Amp., Mm.	n	[illegible]	Temp., C.
953	0.08 .20 .44 .58 .80	13.056 13.053 13.046 13.044 13.037	3/1	23°.0
1,383	0.09 .47 .75	10.836 10.828 10.824	4/1	23°.2
1,814	0.05 .12 .21 .66 .98	9.450 9.449 9.447 9.442 9.439	5/1	24°.4
2,244	0.07 .22 .37 .45 .68	8.480 8.479 8.477 8.476 8.473	5/1	24°.5
2,675	0.08 .10 .16 .23 .44 .79 .81	7.754 7.753 7.753 7.753 7.750 7.746 7.745	5/1	24°.6
3,097	0.14 .32 .44 .60 .62 .68 .78	7.190 7.189 7.187 7.184 7.183 7.182 7.181	6/1	24°.6

III. Since $1/n \propto \sqrt{e}$, one half the correction in per cent. which was made for e, was made here for n: or we might use a formula similar to equation (5)

$$n_t = n_{23}[1 - 0.00045(t - 23)]. \tag{6}$$

For the other three wires the correction was neglected for reasons already stated.

TABLE VII.

Copper.						Steel.			
M[1] Gm.	n	Temp.	Corr.	n_{23}.	Mn^2.	M Gm.	n	Temp.	Mn^2.
956	17.710	23°.7	+0.006	17.716	3.001×10^5	954	9.525	21°.5	8.655×10^4
1,386	14.732	23°.8	+0.006	14.738	3.011	1,384	7.908	21°.6	8.655
1,817	12.865	24°.0	+0.006	12.871	3.010	1,815	5.903	21°.7	8.649
2,247	11.573	24°.0	+0.005	11.578	3.012	2,245	6.199	23°.1	8.627
2,678	10.607	21°.7	−0.005	10.602	3.010	2,676	5.675	23°.0	8.618
3,100	9.841	23°.8	+0.004	9.845	3.005×10^5	3,098	5.268	22°.7	8.598×10^4

Phosphor-bronze.				Platinum-iridium.			
M Gm.	n	Temp.	Mn^2.	M Gm.	n	Temp.	Mn^2.
953	7.767	21.3	5.749×10^4	953	13.058	23.0	1.625×10^5
1,383	6.437	21.8	5.731	1,383	10.837	23.2	1.624
1,814	5.611	21.9	5.711	1,814	9.450	24.4	1.620
2,244	5.038	22.0	5.696	2,244	8.481	24.5	1.614
2,675	4.594	21.3	5.646×10^4	2,675	7.755	24.6	1.609
				3,097	7.193	24.6	1.602×10^5

In Table VII. all the wires but the copper show a steady decrease in Mn^2 with increase of M. The decrease is very small, or zero, between the first two masses, and it has been noted that there was, in general, no decrease of M/e in Table III., until the second mass was reached.

Relation between M and e.

For all except the small masses the results of both the direct measurement of elongation and of the timing, indicate for steel, phosphor-bronze, and platinum-iridium a greater elongation than that demanded by Hooke's Law. We will therefore assume the relation

$$M = \alpha e + \beta e^2, \tag{7}$$

and deduce the period which would result from oscillations along the curve represented by this equation. Since M/e decreases with increasing M, β will evidently be negative. It is more logical, perhaps, to write

$$e = aM + bM^2,\text{[2]} \tag{8}$$

but for solution of the differential equation given later it is simpler to have M expressed in terms of e, as in (7).

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[1] M includes one third the mass of the wire, which addition was negligible for all except copper.

[2] J. O. Thompson found the formula $e = aM + bM^2 + cM^3$ to satisfy his observations in static experiments with wires. See Amer. Jour. Science, Vol. 43, p. 32, 1892.

IV. Derivation of the Period.

If we assume small vibrations of the wire, the period, neglecting the damping term, is known to be

$$T = 2\pi\sqrt{\frac{M}{\frac{dM}{de}g}}. \tag{9}$$

From (7) $dM/de = \alpha + 2e\beta$, whence

$$n = \frac{1}{2\pi}\sqrt{\frac{g(\alpha + 2\beta e)}{M}}. \tag{10}$$

Since the period has been shown by Figs. 8 and 9 to depend on the amplitude, the following derivation is given to show whether vibration along the curve represented by equation (7) causes the period to vary appreciably with the amplitude or not.

Returning to Fig. 1, let x be the distance from O to P at the time t. Let M' be the mass which by equation (7) corresponds to the elongation x. Let F' represent the force of restitution at the time t. M' will be less than M when x is less than e, and greater than M if x is greater than e. Then from equation (7)

$$F' = M'g = Mg\frac{M'}{M} = Mg\frac{\alpha x + \beta x^2}{\alpha e + \beta e^2}. \tag{11}$$

In equation (3) let $h = \frac{\Delta l}{\Delta p} = \frac{AT\alpha^2 L}{\omega c}$

$$\Delta l = \Delta x$$
$$\Delta p = M' - M = \alpha(x - e) + \beta(x^2 - e^2).$$

Δx is the increase in length due to the shifting of the position of O by temperature changes in the wire, the temperature changes being caused by the vibration. Then

$$\Delta x = h\,\Delta p = h[\alpha(x - e) + \beta(x^2 - e^2)]. \tag{12}$$

Since equation (7) does not contain this temperature effect, we must add Δx to the x of that equation, and, instead of x, write $x + h[\alpha(x - e) + \beta(x^2 - e^2)]$, Δx is negative when $x < e$. $\Delta x = 0$ when $x = e$. Thus we take the temperature of the wire to be that of the room when $x = e$. We will also suppose the vibrations to be so rapid that there is no time for the wire to receive or to give out heat, but the vibrations take place under adiabatic conditions. Then we have, for the elastic force acting on the mass M, when the distance OP is x

$$F' = Mg\frac{\alpha\{x + h[\alpha(x - e) + \beta(x^2 - e^2)]\} + \beta\{x + h[\alpha(x - e) + \beta(x^2 - e^2)]\}^2}{\alpha e + \beta e^2}. \tag{13}$$

Since the effect of the weight of the wire itself on the vibrations is equivalent to the effect of one third such weight suspended from the lower end, in what follows, M will denote the suspended mass plus one third the mass of the wire. Except for the copper wire, however, this addition was less than ½ gram, and was neglected.

Let

$$F = F' - Mg. \tag{14}$$

Then if we neglect terms of the order of $h\alpha\beta(x^2 - e^2)$ and $h\alpha\beta x(x - e)$, which are small compared with αx and $\alpha^2 h(x - e)$ we shall have from (13) and (14)

$$\begin{aligned} F = Mg\frac{\alpha(1 + h\alpha)(x - e) + \beta(x^2 - e^2)}{\alpha e + \beta e^2} &= \\ g[\alpha(1 + h\alpha)(x - e) + \beta(x^2 - e^2)]&. \end{aligned} \tag{15}$$

The external and the internal friction are each assumed proportional to the velocity, and the two terms representing them may be combined thus,

$$\mu\frac{dx}{dt} + \eta\frac{dx}{dt} = \sigma\frac{dx}{dt}, \tag{16}$$

where μ is the external damping coefficient, and η the internal.

As the mass M moves up and down, O (Fig. 1), the point to which the lower end of the wire tends to return, will also oscillate, because of the after-effect. This motion of O will doubtless be somewhat behind the motion of P in phase. The complete vibration requiring only about 1/5 sec. for the lowest frequency observed, there would be only a short time for the wire to recover between vibrations, but immediately after release the motion of O due to the after-effect is comparatively rapid, so it is not at all certain that this motion may be disregarded; in fact, a later discussion will indicate that this is perhaps an important factor in determining the period.

Kohlrausch proposed the formula $-dy/dt = \alpha(y/t)$, or $y = c/t^\alpha$, where y is the deformation due to the after-effect at the time t after the distorting force has ceased to act, and α is a constant. If this were approximately true for small t, we should have a very large dy/dt, but all his observations were taken after 10 sec. and as far as the writer has observed, practically nothing is known of the behavior of the after-effect during the first second after removal of the distorting force. Since an assumption would be worth little without the support of observational data and also since it would doubtless greatly complicate the equation, this effect will for the present be neglected.

The differential equation of motion may then be written

$$\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + g\frac{\beta x^2}{M} + \frac{g\alpha(1 + h\alpha)}{M}x - \frac{g\alpha^2he + gM}{M} = 0. \tag{17}$$

If we neglect h and β, then $M = \alpha e$, and (17) becomes

$$\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + \frac{g}{e}(x - e) = 0, \tag{18}$$

which is the well-known equation representing damped vibrations, the solution being

$$x = e + Ae^{-\sigma t/2}\cos\sqrt{\frac{g}{e} - \frac{\sigma^2}{4}}\cdot t, \tag{19}$$

where the time is counted from the lowest point of the vibration, at which point $x = e + A$, A being the amplitude. The period of this motion is

$$T = \frac{2\pi}{\sqrt{\frac{g}{e} - \frac{\sigma^2}{4}}}. \tag{20}$$

If β is neglected, but h is not, we have

$$T = \frac{2\pi}{\sqrt{\frac{g(1 + h\alpha)}{e} - \frac{\sigma^2}{4}}}. \tag{21}$$

From measurements of σ discussed later it will appear that for the largest value of σ found, $\sigma^2/4$ compares with g/e about as 1 with 10^6, showing that under the assumption that the frictional resistance varies as the velocity, the damping has no appreciable effect on the period. Hence, if in (17) we neglect $\sigma(dx/dt)$, we have left

$$\frac{d^2x}{dt^2} + \frac{g}{M}[\beta x^2 + \alpha(1 + h\alpha)x - (\alpha^2he + M)] = 0. \tag{22}$$

This equation represents undamped vibrations, but since we desire only the period that fact is of little importance.

Solution of the Equation.

Let

$$\frac{\beta}{3} = \gamma$$

$$\frac{\alpha(1 + h\alpha)}{2} = \delta$$

$$M + \alpha^2he = \epsilon$$

and (22) becomes

$$\frac{d^2x}{dt^2} + \frac{g}{M}(3\gamma x^2 + 2\delta x - \epsilon) = 0. \tag{23}$$

By a first integration we obtain

$$\frac{p^2}{2} = -\frac{g}{M}\gamma\left(x^3 + \frac{\delta}{\gamma}x^2 - \frac{\zeta}{\gamma}x + D\right), \tag{24}$$

where $p = dx/dt$, and D is the constant of integration.

$$p = \frac{dx}{dt} = \sqrt{\frac{-2g\beta}{3M}}\sqrt{x^3 + \frac{\delta}{\gamma}x^2 - \frac{\zeta}{\gamma}x + D}. \tag{25}$$

D may be determined from the condition, that when $x = e - A$, $p = 0$.

If we let $f(x) = x^3 + (\delta/\gamma)x^2 - (\zeta/\gamma)x + D$ then $f(x) = 0$ when $p = 0$,

$$D = -(e - A)^3 - \frac{\delta}{\gamma}(e - A)^2 + \frac{\zeta}{\gamma}(e - A), \tag{26}$$

and is thus a function of the amplitude.

If $A = e$, which is the greatest amplitude possible, $D = 0$.

This amplitude is always impracticable on account of the tendency of the wire to vibrate transversely when this point is nearly reached.

Equation (25), solved for t, is

$$t = \sqrt{\frac{3M}{-2g\beta}}\int_{x_0}^{x}\frac{dx}{\sqrt{x^3 + \frac{\delta}{\gamma}x^2 - \frac{\zeta}{\gamma}x + D}}. \tag{27}$$

Equation (27) gives the time for any part of the vibration, according to the limits assigned to x. If the extreme positions of P are taken as limits the value of t will be ½T.

To reduce the elliptic integral to the standard form of the first class,

$$F(k, \varphi) = \int_0^{\phi}\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = \int_0^{x}\frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}},$$

we must find the roots of $f(x) = 0$.

Since the 4th power of x is lacking, we have at once that one root is ∞, and D has been chosen so that another root is $e - A$. If we substitute the value of D from equation (26) in $f(x) = 0$, and divide by $x - (e - A)$, we obtain

$$x^2 + x\left(e - A + \frac{\delta}{\gamma}\right) + (e - A)^2 + \frac{\delta}{\gamma}(e - A) - \frac{\zeta}{\gamma} = 0, \tag{28}$$

whence

$$x = -\frac{e - A + \frac{\delta}{\gamma}}{2} \pm \frac{1}{2}\sqrt{(e - A)^2 + 2\frac{\delta}{\gamma}(e - A) + \frac{\delta^2}{\gamma^2} - 4(e - A)^2 - 4\frac{\delta}{\gamma}(e - A) + 4\frac{\zeta}{\gamma}}. \tag{29}$$

ζ may be expressed in terms of δ and γ.

$$\begin{aligned}\zeta = M + \alpha' he &= ae + 2e^2 + a'he \\ &= ae(1 + ha) + 2e^2 \\ &= 2\delta e + 3\gamma e^2\end{aligned}$$

$$4\frac{\zeta}{\gamma} = 8\frac{\delta e}{\gamma} + 12e^2. \tag{30}$$

Substitute this value of $4\frac{\zeta}{\gamma}$ in (29) and the equation becomes

$$x = \frac{1}{2}\left(-e + A - \frac{\delta}{\gamma} \pm \sqrt{9e^2 + A^2 + \frac{\delta^2}{\gamma^2} + 6eA + 6\frac{\delta}{\gamma}e + 2\frac{\delta}{\gamma}A - 4A^2}\right). \tag{31}$$

The expression under the radical differs from the square of $3e + A + \delta/\gamma$ only by $4A^2$. For the largest value of A observed

$$\begin{aligned}\frac{4A^2}{\frac{(\delta)^2}{(\gamma)^2}} &= \frac{1}{90000} \text{ for platinum-iridium,} \\ &= \frac{1}{250{,}000} \text{ for phosphor-bronze,} \\ &= \frac{1}{640{,}000} \text{ for steel.}\end{aligned}$$

For very small amplitudes the ratio is much smaller. Hence $4A^2$ may be neglected and we may write

$$x = \tfrac{1}{2}\left[-e + A - \frac{\delta}{\gamma} \pm \left(3e + A + \frac{\delta}{\gamma}\right)\right]. \tag{32}$$

$$\therefore \left\{\begin{aligned} x_3 &= e + A \\ x_4 &= -2e - \frac{\delta}{\gamma} \end{aligned}\right\} \tag{33}$$

are the remaining two roots of $f(x) = 0$.

It was to be expected that if $e - A$ is one root, $e + A$ is very nearly another, which simply means that in the vibration P moves very nearly as far below the point $x = e$ as it does above that point.

If we assume $e - A$ and $e + A$ to be two of the roots, the third root comes easily from the relation between the roots and the coefficients; for if ω_1, ω_2 and ω_3 are the roots, and

$$\begin{aligned}\omega_1 &= e - A, \\ \omega_2 &= e + A,\end{aligned}$$

then

$$\Sigma\omega = -\frac{\delta}{\gamma}. \tag{34}$$

and

$$\omega_3 = -2e - \frac{\delta}{\gamma}.$$

To obtain the integral in (27) in the standard form we have to transform the roots

$$\infty,\ e - A,\ e + A,\ -\frac{\delta}{\gamma} - 2e$$

respectively into

$$-\frac{1}{k},\ -1,\ 1,\ \frac{1}{k}.$$

The bilinear transformation is

$$x = \frac{py + r}{qy + s}.$$

Determining the constants, we find

$$\begin{aligned} q &= 1, \\ s &= \frac{1}{k}, \\ p &= e + \frac{A}{k}, \\ r &= \frac{e}{k} + A. \end{aligned} \tag{35}$$

If $A = e$, $p = r$.

If $A = 0$, $k = 0$, and $s = r = \infty$.

$$\int_{x_0}^{x_1} \frac{dx}{\sqrt{f(x)}} = \frac{ps - qr}{\sqrt{M'}} \int_{y_0}^{y_1} \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}}.$$

$$M' = \frac{\sqrt{b_0}}{k},\quad b_0 = q^4 f\left(\frac{p}{q}\right),\ = f(p),\ \text{for } q = 1;$$

$$\therefore\ \frac{ps - qr}{\sqrt{M'}} = \frac{e + \dfrac{A}{k} - kA - e}{\sqrt{f(p)}} = \frac{\dfrac{A}{k} - kA}{\sqrt{f(p)}}, \tag{36}$$

$$k = \frac{1 - \sqrt{a}}{1 + \sqrt{a}},\ \text{where } a = \frac{\omega_4 - \omega_3}{\omega_4 - \omega_2} \cdot \frac{\omega_1 - \omega_2}{\omega_1 - \omega_3}. \tag{37}$$

The three finite roots are all positive, and the graph of the function appears somewhat as in Fig. 10. The integration takes place from $\omega_2 = e - A$ to $\omega_3 = e + A$, and is therefore real.

D is negative except when $A = e$ in which case it is zero.

$$a = \frac{\dfrac{\delta}{\gamma} + 3e + A}{\dfrac{\delta}{\gamma} + 3e - A}. \tag{38}$$

Let $c = -(\delta/\gamma) - 3e$, a positive quantity. Then

$$k = \frac{1 - \sqrt{\dfrac{c-A}{c+A}}}{1 + \sqrt{\dfrac{c-A}{c+A}}} = \frac{c - \sqrt{c^2 - A^2}}{A}. \tag{39}$$

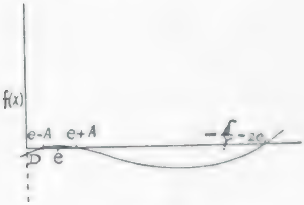

Fig. 10.

When $A = 0$, $k = 0$, and the last member of equations (36) becomes indeterminate. To evaluate it, from (39)

$$\frac{A}{k} = \frac{A}{\dfrac{c - \sqrt{c^2 - A^2}}{A}} = c + \sqrt{c^2 - A^2}. \tag{40}$$

hence,

$$\lim_{A=0} \frac{A}{k} = 2c, \tag{41}$$

also from (39),

$$Ak = c - \sqrt{c^2 - A^2}; \tag{42}$$

$$\therefore \frac{A}{k} - kA = 2\sqrt{c^2 - A^2} \tag{43}$$

and

$$\frac{ps - qr}{\sqrt{M'}} = \frac{2\sqrt{c^2 - A^2}}{\sqrt{f(p)}}; \tag{44}$$

$$\begin{aligned} \therefore t = \sqrt{\frac{3M}{-2g\beta}} \int_{x_0}^{x_1} \frac{dx}{\sqrt{f(x)}} = \frac{2\sqrt{c^2 - A^2}}{\sqrt{f(p)}} \\ \times \sqrt{\frac{3M}{-2g\beta}} \int_{y_0}^{y_1} \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} \end{aligned} \tag{45}$$

When $x_0 = e - A$, $y_0 = -1$; when $x_1 = e + A$, $y_1 = 1$. Integrating between these limits we have $\frac{1}{2}T$. Hence the period is given by

$$T = \frac{8\sqrt{c^2 - A^2}}{\sqrt{f(p)}} \sqrt{\frac{3M}{-2g\beta}} K. \tag{46}$$

where K is the value of the complete elliptic integral

$$\int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}.$$

k was found to be very small for all values of A really obtained, and it approaches zero with A. Thus if $A = 0.5$ cm. for platinum-iridium, $\sin^{-1} k = 22'$. If $A = 1.0$ cm. for steel $\sin^{-1} k \doteq 7'$. But the largest value of A given in Figs. 11 and 12 is about 0.1 cm. This gives for platinum-iridium $\varphi = \sin^{-1} k = 4'\ 21''$ and for steel $\varphi = 0'\ 42''$. For phosphor-bronze φ is slightly larger than for steel. If $k = 0$,

$$\int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = \int_0^1 \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2} = 1.5708.$$

If $\sin^{-1} k = 1°$,

$$\int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = 1.5709.$$

Since k is in no case larger than $\sin 30'$, and for ordinary amplitudes is much smaller, we may use $K = \pi/2$ for all the wires and all amplitudes.

The effect of varying amplitude on the other factors in eq. (46) will be shown after α and β have been determined.

Simplification of the Equation (46) for Small Amplitudes.

When $A = 0$, (46) becomes

$$T = 2\pi \frac{2c}{\sqrt{f(p)}} \sqrt{\frac{3M}{-2g\beta}}. \qquad (47)$$

This may be further simplified,

$$D = -f(e), \text{ when } A = 0, \qquad (48)$$

$$\frac{\zeta}{\gamma} = 2\frac{\delta}{\gamma}e + 3e^2,$$

$$\begin{aligned} f(p) &= p^3 - e^3 + \frac{\delta}{\gamma}(p^2 - e^2) - \frac{\zeta}{\gamma}(p - e) \\ &= (p-e)\left[p^2 + pe + e^2 + \frac{\delta}{\gamma}p + \frac{\delta}{\gamma}e - 2\frac{\delta}{\gamma}e - 3e^2\right] \\ &= (p-e)\left[p^2 + pe - 2e^2 + \frac{\delta}{\gamma}p - \frac{\delta}{\gamma}e\right], \end{aligned}$$

$$f(p) = (p-e)^2\left(p + 2e + \frac{\delta}{\gamma}\right). \qquad (49)$$

From equation (35)

$$p = e + \frac{A}{k} = e + 2c, \text{ when } A = 0; \qquad (50)$$

$$\therefore f(p) = (2c)^2(e + 2c - e - c) = 4c^3. \qquad (51)$$

Hence, from (47),

$$T = 2\pi\frac{1}{\sqrt{c}}\sqrt{\frac{3M}{-2g\beta}} = 2\pi\sqrt{\frac{3M}{-2g\beta\left(-\frac{\delta}{\gamma}-3e\right)}},$$

$$T = 2\pi\sqrt{\frac{M}{g[\alpha(1+h\alpha)+2\beta e]}}. \qquad (52)$$

$$n = \frac{1}{2\pi}\sqrt{\frac{g[\alpha(1+h\alpha)+2\beta e]}{M}}. \qquad (53)$$

This is the final form for the expression of the frequency for vibrations of small amplitude, in which the motion is along the curve represented by the equation $M = \alpha e + \beta e^2$, except for the small heating effect indicated by the term containing h. The effect of larger amplitudes as shown by (46) will be mentioned later.

Omitting the term $h\alpha$, (53) is the same as (10). If the wire obeyed Hooke's Law, we would have only to set $\beta = 0$, and (53) reduces to

$$n = \frac{1}{2\pi}\sqrt{\frac{g\alpha(1+h\alpha)}{M}} = \frac{1}{2\pi}\sqrt{\frac{g(1+h\alpha)}{e}}, \qquad (54)$$

which is equivalent to (21) except for the damping term.

Computation of Constants for the Equation $M = \alpha e + \beta e^2$.

The next step is to find the constants α and β of the equation $M = \alpha e + \beta e^2$, which will best satisfy the observed values of the frequency as given by equation (53).

We have from (53)

$$Mn^2 = \frac{1}{4\pi^2}[g\alpha(1+h\alpha)+2\beta e].$$

Since $2\beta e$ is small compared with $g\alpha(1+h\alpha)$ it will be sufficient to write M/α in place of e; for $e = M/\alpha - \beta e^2/\alpha$, and the maximum value of $\beta e^2/\alpha$ is about one per cent. of M/α. This will influence the determination of β only, and one per cent. in the value of β is negligible. Making this substitution for e in (53), the equation becomes

$$Mn^2 = \frac{1}{4\pi^2}\left[g\alpha(1+h\alpha)+\frac{2\beta}{\alpha}M\right]. \qquad (55)$$

If we plot Mn^2 as a function of M, we obtain for steel, phosphor-bronze, and platinum-iridium the results shown in Fig. 11. The slope of each of these lines is $\beta/2\pi^2\alpha$ for that wire, and the intercept on the Mn^2 axis is $g\alpha(1+h\alpha)/4\pi^2$. $h\alpha$ is of the order 0.001 for all the three wires to which the equation applies, hence for $h\alpha$ we may use the value obtained by neglecting h and β altogether. The values of $1 + h\alpha$ are

Copper	1.0021
Steel	1.0013
Phosphor-bronze	1.0015
Platinum-iridium	1.0011

The heating effect has therefore little influence on the frequency.

It will be noticed on the steel and platinum-iridium plots that when M is small Mn^2 is less than it should be. This may be explained by supposing that the wire obeys Hooke's Law for small elongations. Also, as will appear later in comparing the results for direct elongation, the wires seem not to be entirely straightened out until the second mass is reached, in spite of the precautions, already noted, which were taken to straighten them. The phosphor-bronze wire does not exhibit this deficiency in Mn^2 because the elongation of that wire was relatively greater, and the first weight took the wire beyond the region where the elongation is proportional to the load. The largest weight used was too great for the phosphor-bronze wire, and although no permanent elongation could be observed when this mass was removed, yet it is plainly evident from the

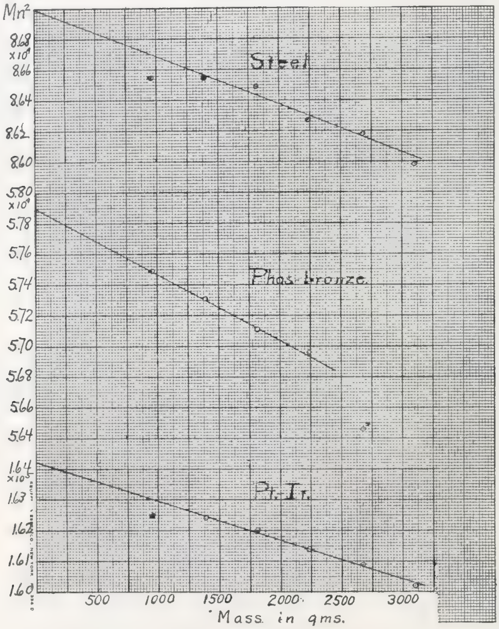

Fig. 11.

small value of Mn^2 for this mass that the rate of elongation of the wire at this point was greater than that which is given by $M = \alpha e + \beta e^2$.

For the copper wire, Mn^2 is a constant within the range of error, but it is to be noted that the greatest elongation is only about the same as the range for which Hooke's Law holds in the case of the steel and platinum-iridium wires. If there is any range for which equation (7) represents the behavior of the copper wire, that range is very limited, for the product of Mn^2 is a constant as far as observed, and the wire could sustain only a little more than the heaviest weight without passing the elastic limit.

From Fig. 11 the following equations were obtained.

$$\text{Steel: } M = 3{,}498\, e - 22.0\, e^2. \tag{56}$$

$$\text{Phosphor-bronze: } M = 2{,}327\, e - 19.6\, e^2. \tag{57}$$

$$\text{Platinum-iridium: } M = 6{,}602\, e - 166\, e^2. \tag{58}$$

To show with what degree of accuracy these represent the observed frequencies, we write eq. (53) in the form

$$n = \frac{1}{2\pi}\sqrt{\frac{g\alpha(1 + h\alpha)}{M} + 2\frac{\beta}{\alpha}g}. \tag{59}$$

and compute n for each mass, comparing the observed with the computed values.

TABLE VIII.

Steel.				Phosphor-bronze.				Platinum-iridium.			
M Gm.	n_0.	n_c.	$n_0 - n_c$.	M Gm.	n_0.	n_c.	$n_0 - n_c$.	M Gm.	n_0.	n_c.	$n_0 - n_c$.
954	9.525	9.531	−0.006	953	7.767	7.766	+0.001	955	13.058	13.076	−0.018
1,384	7.908	7.908	.000	1,383	6.437	6.436	+0.001	1,383	10.837	10.836	+0.001
1,815	6.903	6.900	+ .003	1,814	5.611	5.611	.000	1,814	9.439	9.440	−0.001
2,245	6.199	6.199	.000	2,244	5.038	5.047	− .009	2,244	8.481	8.479	+0.002
2,676	5.675	5.673	+ .002	2,675	4.594	4.606	− .012	2,675	7.750	7.752	−0.002
3,098	5.268	5.269	−0.001					3,095	7.193	7.193	0.000

n_0 = observed value of the frequency for infinitely small vibrations, taken from Fig. 9 for platinum-iridium.

n_c = frequency computed from equation (59).

The above residuals show that the frequency is very well represented by equation (59) except for the first masses of steel and platinum-iridium and the last of phosphor-bronze. For the smallest masses it is not certain that the wires were fully straightened, but on examining the results for all the wires including the copper we must conclude that for small distortions, the elongation is proportional to the distorting force, beyond that there is a region where the wire very closely obeys the law expressed by equation (7), then the elongation increases more rapidly

TABLE IX.

Copper.					Phosphor-bronze.					Platinum-iridium.				
M[1] Gm.	e_0 Cm.	n_0.	n_e.	n_0-n_e.	M gm.	e_0 Cm.	n_0.	n_e.	n_0-n_e.	M Gm.	e_0 Cm.	n_0.	n_e.	n_0-n_e.
956	0.0822	17.716	17.397	+0.319	953	0.4138	7.767	7.752	+0.015	953	0.1461	13.058	13.044	+0.014
1,386	0.1184	14.738	14.496	.242	1,383	.6023	6.437	6.426	+0.011	1,383	.2119	10.837	10.831	+0.006
1,817	0.1563	12.871	12.617	.254	1,814	.7912	5.611	5.606	+0.005	1,814	.2780	9.450	9.456	−0.006
2,247	0.1945	11.578	11.310	.268	2,244	.9816	5.038	5.033	+0.005	2,244	.3448	8.481	8.491	−0.010
2,678	0.2320	10.602	10.356	.246	2,675	1.1742	4.594	4.602	−0.008	2,675	.4137	7.755	7.752	+0.003
3,100	0.2695	9.845	9.609	+0.236						3,097	.4812	7.193	7.187	+0.006

---

[1] M does not include the mass of the scale-pan, which was 223 gm. e_0 is the change in elongation due to M. For the vibrations, M is the tota mass suspended.

even than this formula demands, and finally the strain becomes so great that the wire gives way. The weaker parts of the wire pass through these stages in advance of the stronger parts, and their cross-section being lessened, they elongate more and more rapidly, until rupture takes place at one of these points.

Computation of Frequency for Wires Obeying Hooke's Law.

Formula (21) gives the period if the wire obeys Hooke's Law and if we measure the elongation directly. For small elongations we might expect this formula to give a period agreeing well with the observed, but it does not. In all cases the e observed directly and substituted in (21) gives too small values for n, indicating that the observed e is too large.

Table IX gives the values of n_e computed from formula (21), using the values of e observed directly. The masses are the same as those used in the vibrations, except for steel. For this reason steel is given separately in Table X.

At first sight it might be supposed that these results for phosphor, bronze and platinum-iridium are satisfactory, with the exception, perhaps, of the smallest mass; and that formula (21) gives the period more closely than (59): but it must be remembered that formula (21) assumes that $e \propto M$. Therefore we have no right to substitute in it the values of e_0 in Table IX, because the ratio e_0/M is not constant, and we need to use a different ratio of e_0/M for each one, which is contrary to the hypothesis on which (21) is derived.

TABLE X.

Steel.

M Gm.	$\frac{e}{M}$.	M Gm.	e Cm.	n_0.	n_e.	$n_e - n_0$.
1,000	2.874×10^{-4}	954	0.2754	9.525	9.501	+0.024
1,500	2.873	1,384	.3996	7.908	7.888	+0.020
2,000	2.883	1,815	.5240	6.903	6.888	— .015
2,500	2.885	2,245	.6481	6.199	6.193	+ .006
3,000	2.892	2,676	.7726	5.675	5.673	+ .002
Mean...	2.887×10^{-4}	3,098	.8944	5.268	5.272	— .004

Columns 3 and 4, Table X, give the vibrating masses and the corresponding elongations, which were computed from the mean e/M by multiplying by the mass used in vibration. n_e is the frequency from

$$n_e = \frac{1}{2\pi}\sqrt{\frac{g(1+hm)}{e}},$$

using the e of column 4. Table X. shows, as the values of Mn^2 of Table VII. also do, that no constant value of M/e will satisfy the observed frequencies.

Comparison of e from Static and Dynamic Observations.

Finally the elongations corresponding to the observed frequencies were computed for all the wires. For the copper wire the formula, $e = g(1 + h\alpha)/4\pi^2 n^2$ was used. For the others formulæ (56), (57), and (58) were inverted, giving e in the form of equation (8).

$$\text{Steel: } e = 2.859 \times 10^{-4} M + 5.1 \times 10^{-10} M^2. \tag{60}$$

$$\text{Phosphor-bronze: } e = 4.297 \times 10^{-4} M + 1.6 \times 10^{-9} M^2. \tag{61}$$

$$\text{Platinum-iridium: } e = 1.515 \times 10^{-4} M + 5.9 \times 10^{-10} M^2. \tag{62}$$

For these three wires e_c was computed for the given load plus the pan, then for the pan alone, and the difference compared with the elongation, e_0, observed directly. In the case of copper since the product of Mn^2 was a constant no computation was made for the initial load.

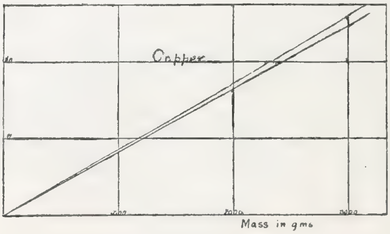

Fig. 12.

Fig. 12 shows e_0 and e_c for copper as ordinates with M on the axis of abscissas. e_c and M give a straight line, while e_0 and M do not. For the other wires neither curve is a straight line, and they lie so near together that they have not been plotted. $e_0 - e_c$ is always positive, however, for all the wires, and increases with M.

V. DISCUSSION OF RESULTS.

The dynamic modulus of metals has been found by other methods to be greater than the static modulus. The same is true for these longitudinal vibrations, as shown in the last paragraph. No attempt has been made to determine the absolute value of either modulus, it being outside the purpose of this investigation.

If we consider the modulus to be proportional to dM/de, we have from equation (7) $dM/de = \alpha + 2e\beta$, which may easily be computed by equations (56), (57) and (58). For copper, since $\beta = 0$, we have $M/e = 4\pi^2 Mn^2/g(1 + h\alpha)$.

Possible After-effect in All the Wires.

A large after-effect is shown in Fig. 5 for copper, and some is also exhibited by similar curves for steel. Since the recovery is rapid at first, it seems quite possible that the point of instantaneous recovery may be far beyond that indicated by the dotted continuation of the curves. The greater part of the recovery may be during the first second or two following release, or even while the load is being removed. The curves for copper unquestionably show that it is rapid at first, for very little of the effect there indicated comes from the heating of the wire. For phosphor-bronze and platinum-iridium there is *apparently* no after-effect, but the recovery may be so rapid that it is complete before an observation can be taken. This theory would explain the discrepancy between the observed elongation and that computed from the results of vibrations, which is equivalent to saying it would account for the difference between the static and dynamic moduli.

We may suppose, then, that the observed values of the elongation are all too large. The deformation due to the after-effect is much larger than would be supposed from usual observation, for it decreases very rapidly when the load is first removed, and by the time an observation can be made a great part of it has disappeared. Thus there may be considerable after-effect in wires which apparently show none; the wire may have entirely recovered by the time the first observation can be made.

Influence of the After-effect on the Period.

Considering again Fig. 1, as the point P oscillates, O will also oscillate with a certain amplitude depending on the period, the distance OP, and the material of the wire, as well as its condition as to hardness, temperature, etc. The phase difference between the motions of O and P will depend on the period and the manner in which the wire recovers after removal of the load. This last is an important element, and also an uncertain one. If the phase difference were zero, then P and O would both move downward through their mean positions at the same instant. In the half of the period when P is below its mean position, O would also be below its mean position. As a result the force of restitution would be less than if O remained at rest, and the period would be increased. When P and O were above the mean positions there *would* be a greater elastic tension on account of the displacement of O, and this would also increase the period, so that we have a decrease in frequency due to the motion of O. It may be noted that this is just opposite to the effect of the motion of O due to the heating and cooling of the wire. (Comp. the term $h\alpha$ in equation (53).)

But we must suppose that O is somewhat behind P in phase. Then when the two points are on opposite sides of their respective positions of rest the frequency would tend to increase, when they are on the same side it would tend to decrease. Unless the motion of O became 90° or more behind that of P, the frequency would probably be decreased by this motion of O.

That this does not account for the variation in Mn^2 for steel, phosphor-bronze, and platinum-iridium, appears from the fact that Mn^2 did not vary appreciably in the case of copper, which is just the one in which the after-effect was greatest. It is true that the recovery of copper is not so rapid as that of the other three metals, and hence the amplitude of O may not be as great, but we should expect at least a noticeable variation in Mn^2.

VI. The Damping of Longitudinal Vibrations.

Apparatus.

The amplitude of the vibrations when the wire is only about two meters long cannot be very great; consequently the accurate measurement of the damping becomes a matter of some difficulty. The following method has yielded fairly satisfactory results.

A fiber of glass was attached to the wire, and allowed to project about 1 cm. at right angles to it. The fiber pointed towards a specially constructed camera. The lens, which was one used in an ordinary galvanometer telescope, and had a focal length of about 15 cm., was stopped down until the aperture was only about 5 mm. At the back of the camera was fixed a drum, 40 cm. in circumference, rotating about a vertical axis. The axis was a screw, so that as the drum revolved, it moved parallel to the axis. A small drop of mercury was placed on the end of the glass strip next to the camera, the drop was brightly illuminated by the light from an electric arc, and the camera was so placed that the image of the drop fell on the drum. Owing to the great curvature of the small mercury drop, a sharp spot of light was obtained. A rapid photographic film was attached to the drum, and rotated as the wire vibrated, thus obtaining a record of the vibration. The mercury drop could not be placed directly on the wire, because of the fogging of the film by light reflected from the wire. The glass fiber was colored red, so that any light reflected therefrom would not affect the record. Fig. 13 shows a section of a record for steel.

Since the purpose was only to measure the amplitude, uniformity of motion of the drum was unnecessary. It was therefore turned by hand, a handle outside the camera box serving that purpose. The mass was set vibrating and the drum started. Then the current driving the wire

Fig. 1A. [illegible]

GEORGE A. LINDSAY.

was shut off and at the same time the drum was stopped for an instant. The several vibrations thus heaped up together indicated where the damping began.

The measurements of the amplitude were made on a measuring engine of the Detroit Observatory, the plate-holder of the engine being mounted to revolve about an axis parallel to the axis of the microscope. This permitted an easy adjustment of the cross hair tangent to two adjacent waves. For all the wires except copper, 1,000 vibrations could easily be measured. Beyond that number they were so small that a relatively large error in the logarithm of the amplitude was introduced. The lens was so placed that the amplitude on the film was about equal to the actual amplitude of the point on the wire. The motion might easily have been magnified, but what was gained in this way was lost in definition of the trace. The largest double amplitude measured was about 4 mm. on the film, the smallest about 0.3 mm. Ordinarily the measurements ran from 3 mm. to 0.5 mm. The measuring engine read directly to 0.005 mm.

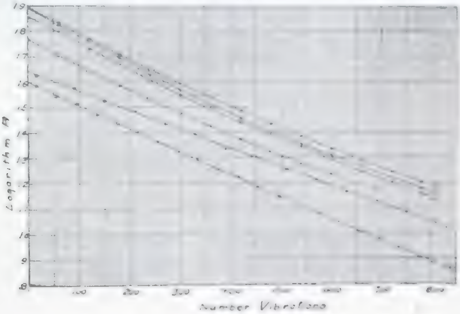

Fig. 14.

Decrement curves for steel.

Two records were made for each mass suspended from the copper wire, because the rapid damping allowed room for two on the film. For the other metals the damping was so small that only one record was made on each film. Every 30th vibration was measured and the logarithms of these were averaged by pairs, so that the plots show only the logarithms for every 60th vibration. Figs. 14 and 15 represent graphically the decrements for the masses given in Table XI. in order of increasing magnitude. Similar curves were drawn for copper, and the logarithmic decrements of Table XI. were obtained from these curves.

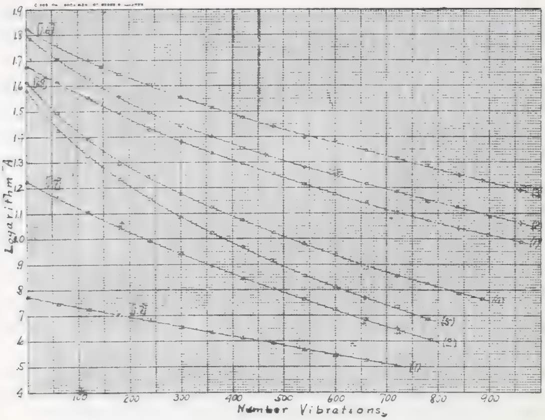

Fig. 15.

Decrement curves, ⊙, for platinum-iridium; △, for phosphor bronze. The scale is the same for all; but to avoid confusion the origin of ordinates has been shifted up or down for certain curves, as indicated by the logarithms in brackets.

Variation of the Logarithmic Decrement.

It was found by Schmidt[1] for torsional, and later also by Voight[2] for flexural and torsional vibrations, that the logarithmic decrement varied with the amplitude. They expressed the relation in the form

$$\lambda = \lambda_0 + bA^2,$$

where λ_0 is the logarithmic decrement for very small vibrations.

The values of Table XI. for copper are plotted in Fig. 16. Although the rate of increase is different for different masses, the increase is nearly proportional to the increase in amplitude[3] in all cases, and

$$\lambda = \lambda_0 + bA$$

represents the relation. The second mass gave an abnormally large decrement for the larger amplitudes, as seen from Table XI. There is no apparent reason for this. Both records taken of this mass show the same peculiarity.

While with the other wires, the logarithmic decrement for a rough approximation, increased in proportion to the increase of amplitude, the increase was much less than with copper, and less regular. The logarith-

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[1] Loc. cit.

[2] Loc. cit.

[3] Bouasse and Carriere found $\lambda = bA$ for torsional vibrations of copper. (See Ann. de Chim. et de Physique, 8me Serie, t. 14, p. 208.)

TABLE XI.

Log. Decr.—Amplitude. Copper.

λ in common logarithms.

Mass. Gm.	Log. A.		1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.0	.9	.8	Temp. C.	n
	L	A	6.31	5.01	3.98	3.16	2.51	1.99	1.58	1.26	1.00	.79	.63		
956	2.30	λ						.00274	217	224	206	179		21°.9	17.716
	1.55									.00222	198	181	169		
1,386	2.75							.00475	384	305	259	226		21.2	14.138
	2.85						.00571	494	377	274	252				
1,817	5.95			.00559	470	394	362	308	277	256	236	219		21.3	12.871
	6.30			.00546	475	410	367	309	277	240	228				
2,247	5.80					.00354	320	294	252	228	206	203		21.9	11.578
	3.95					.00342	329	283	234	219	205	198			
2,678	5.20				.00453	404	386	345	323	293	263	250		21.2	10.602
	5.20				.00454	420	374	339	294	280	264	231			
3,109	4.05					.00403	364	320	296	270	259	221		21.2	9.341
	6.33			.00562	491	436	370	325	292	289	242	240			

Steel.

Mass.	A	A	6.31	5.01	3.98	3.16	2.51	1.99	1.58	1.26	1.00	Temp. C.	n
933	4.00	λ				.00091	90	90	90	88	83	22°.0	9.535
1,363	4.35				.00079	77	74	74	77	80		23.6	7.906
1,803	6.15			.00103	89	93	85	79				23.8	6.963
2,240	8.15		.00102	99	92	90	86	80				22.7	6.109
2,670	8.25		.00114	107	100	93	87	76				22.8	5.625
3,098	7.70			.00110	107	95	91	85	80	76		23.3	5.209

TABLE XI.—*Continued.*

Phosphor-bronze.

M Gm.	A_0	A	6.31	5.01	3.98	3.16	2.51	1.99	1.58	1.26	1.00	Temp. C.	n_0
1,383	6.05			.00040	37							21 .7	6.437
2,244	5.60				.00100	87	78	69	64			21 .8	5.038

Platinum-iridium.

M Gm.	A_0	A	6.31	5.01	3.98	3.16	2.51	1.99	1.58	1.26	1.00	Temp. C.	n_0
953	4.95				.00107	101	82	71	63	58	45	23 .7	13.058
1,383	6.50			.00153	115	93	72	61	57	49		23 .9	10.837
1,814	4.35				.00108	92	77	65	54	52	50	24 .3	9.450
2,240	7.15			.00188	156	126	102	85	76	70	57	24 .2	8.481
2,675	6.70			.00245	190	157	139	115	96	89	75	24 .2	7.755

The column A_0 contains the largest amplitude measured on each film. The last two columns give the temperatures inside the box when the photographs were taken, and the frequencies for small amplitudes.

mic decrement for steel, especially for the smaller masses, is very nearly constant.

For phosphor-bronze only two masses were used and there is a remarkable difference in the decrement in these two cases. The decrement for the small mass, 1,383 gm., is only about one third that for the larger mass, 2,244 gm., and that for the smaller mass is very nearly constant. In 700 vibrations this amplitude decreased from 6.04 to 3.27, while for the larger mass in the same number of vibrations it decreased from 5.56 to 1.48. For the other metals there is no apparent dependence of λ

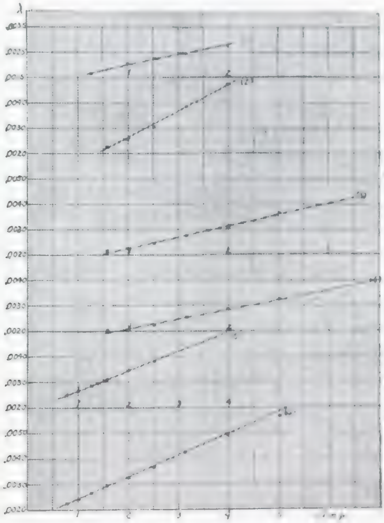

Fig. 16. Copper.

on the period. If we produce the straight lines of Fig. 16, backward to the axis we get for λ_0, omitting No. 2

Mass	λ_0
(1)	.00121
(2)	———
(3)	.00144
(4)	.00130
(5)	.00186
(6)	.00166

With the largest mass suspended from the copper wire a single test was made of the effect of starting with a larger amplitude. In the case of the greater initial amplitude the decrement is slightly higher for large amplitudes, but as the amplitudes decrease the decrements in the two cases approach equality.

The temperatures were nearly the same in all cases, so little variation can be expected from this source.

It would be very desirable to test more fully a single wire, carrying a constant load, to determine the effect of change of temperature, and previous treatment of the wire, as well as the variations of other conditions for which there was not opportunity in this limited investigation.

Cause of Damping.

If the damping were caused by internal friction of the particles, and the friction were assumed proportional to the velocity, then we should have $\lambda \propto \frac{T}{M} \propto n$, or $\lambda/n =$ a const. That this is not fulfilled at all by any of the wires is plainly seen by reference to Table XI., and by comparing λ for equal amplitudes and different frequencies. The phosphor-bronze shows a marked variation in the other direction; λ decreases with increase of frequency.

In Boltzmann's theory the decrement is independent of the period, and this seems to be nearer the truth, for longitudinal vibrations. As already stated in the introduction, Voight found for torsional and flexural vibrations that some metals showed, as phosphor-bronze has done in the present instance, an increase of λ with decrease of n. A definite conclusion can not be reached for phosphor-bronze, however, since but two frequencies were used.

Correction for Air Damping.

A correction for the damping due to the friction of the air was made as follows. The cylindrical shell shown in Fig. 6 with the rod attached as when used for vibrations, was suspended in a horizontal position by a bifilar thread, so as to swing in a direction parallel to the axis, thus exposing the same surface and giving it the same motion through the air. The wire in longitudinal vibration presented only a small surface compared with the shell, and its effect would be nearly compensated by the friction of the bifilar threads mentioned above. The logarithmic decrement of this pendulum system was then determined for the different masses.

From equation (16) $\mu = 2\lambda_0/T_0$, where λ_0 and T_0 are respectively the logarithmic decrement, and period of the pendulum vibrations.

$\eta = 2(\lambda/T - \lambda_0/T_0) = 2\lambda_m/T_m$, where λ_m is the logarithmic decrement due to the internal damping alone. T_m is the period as it would be if there were no air damping, and is practically identical with T.

$$\therefore \lambda_m = \lambda - \lambda_0 \frac{T}{T_0}.$$

Thus $\lambda_0(T/T_0)$ is the correction to be subtracted from the measured value of λ to obtain the decrement due to the metal alone.

For small amplitudes the following were observed.

Mass.	λ_0	T_0 sec.
2,244	0.00027	3.44
3,097	0.00027	3.44

The greatest correction to be applied is for the smallest value of n. The smallest value of n was for phosphor-bronze, where for $M = 2{,}244$ gm., $n = 5.04 \frac{\text{vibs.}}{\text{sec.}}$

There was no mass of 3,097 gm. used with phosphor-bronze, but for steel

$M = 3{,}097$ gm., $n = 5.27$
$M = 2{,}244$ gm., $n = 6.20$

Thus we have for phosphor-bronze,

M gm.	$\frac{\lambda_0 T}{T_0}$.
2,244	0.000019

for steel,

2,244	0.000015
3,097	0.000016

Therefore the maximum correction to be applied to λ given in Table XI. is 0.00002. For the copper it would be about half this value, and hence, when compared with the decrement for internal damping, is small enough to be neglected.

The Platinum-Iridium Wire.

The platinum-iridium wire was chosen because of the peculiar behavior of such wires noticed by Guthe,[1] and later investigated further by the same author and Sieg.[2] They found an excessive damping of torsional vibrations for platinum-iridium wires containing 40 per cent. of iridium, which is the same composition as that of the wire used in the present instance. It was accordingly expected that the damping of the longitudinal vibrations would also be large. The result was a decrement comparable to that of steel and phosphor-bronze, and far below that of the copper. When the wire was later suspended and vibrated torsionally, it was found that neither did this wire exhibit unusually large damping of torsional vibrations. The highest decrement observed for the torsional vibrations was 0.00270, when the initial amplitude was 825°. The decrement decreased considerably with the amplitude. Since the length of the wire under discussion was 145 cm. and the one used by Guthe and Sieg was 40 cm., an angle of 825° is about double the twist per unit length for which they observed a logarithmic decrement some 2½ times as large. The diameter of the wire used here was 0.206 mm., as compared with 0.194 mm. for theirs. Their wire showed a difference in period of 2 per cent. between large and small amplitudes. The period of this wire was found to be 18.069 sec. for 11° amp., 18.133 sec. for 550° amp., a difference of only ⅓ per cent. There is, therefore, evidently a difference in the composition of the two wires, or else previous treatment is very influential in determining their behavior.

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[1] Proc. Iowa Ac. Sci., 15, p. 147, 1908.

[2] Phys. Rev., Vol. XXX., No. 4, 1910.

Variation of Period with Amplitude.

The effect of damping on the period is given by

$$T = \frac{2\pi}{\sqrt{\dfrac{g}{e} - \dfrac{\sigma^2}{4}}},$$

where $\sigma = 2\lambda/T$.

For copper, which exhibited the greatest damping, let us take $\lambda = 0.010$ in natural logarithms, which corresponds to the highest λ computed in common logarithms. Also take $T = 0.07$ sec., which is approximately the period for $M = 1{,}386$ gm. Then

$$\sigma = \frac{.02}{.07} = 0.29, \quad \frac{\sigma^2}{4} = 0.020,$$

$$\frac{g}{e} = \frac{980}{.12} = 8170.$$

Hence $\sigma^2/4$ compares with g/e about as 1 with 400,000. The observed variation of T was ⅓ per cent., hence the variation was not caused by the change in λ, provided σ enters into the equation for the period as shown above. The fact that λ varies so greatly with the amplitude shows however, that the internal friction, whatever may be its nature, is *not* proportional to the velocity.

The equation (46) is an expression for the period which involves the amplitude. The largest value of A observed in timing was 0.1 cm. Substituting this in equation (46) we find that for platinum-iridium, $\sqrt{c^2 - A^2}$ differs from c by 1 part in 700,000, and in the denominator, $\sqrt{f(p)}$ differs, when $A = 0$, and when $A = 0.1$ cm. by only 1 part in 3,000,000. It has already been stated that $K[= F(k, \pi/2)]$ is not perceptibly affected by changing A within the limits of the experiment; hence the vibration along the curve represented by (60), (61), or (62) will not account for the variation of T. This also appears by considering that for copper the dynamic modulus was a constant, yet the greatest variation of period with amplitude was observed here.

By comparing elongations from static and dynamic observations as in Fig. 12 it is found that the differences $e_0 - e_c$ per unit length, in the order of magnitude beginning with the largest, place the metals in the following order: Copper, platinum-iridium, phosphor-bronze, steel.

This is the order of variation of period with amplitude; the copper showed the greatest variation, the platinum-iridium considerable, the phosphor-bronze very little and none was detected for steel.

The position of the wires with regard to the magnitude of λ is uncertain for all except copper, which again undoubtedly stands at the head.

The indication is that the wires having the largest after-effect, have also the greatest variation of period with amplitude, and the largest decrement, although it is to be noted that *rapidity of recovery* from after-effect is of as much importance in affecting the period as the total magnitude of the after-effect.

VII. Summary.

1. As a result of the measurement of the frequency of longitudinal vibrations of wires, carrying various loads, it appears in Section IV. that for small loads, the modulus is a constant; for greater loads the modulus decreases with increasing load, although the wire is still far within the elastic limit.

2. These two regions are of different relative extent in different wires. A soft copper wire showed a constant modulus until the elastic limit was very nearly reached (see Table VII.). Steel, phosphor-bronze and platinum-iridium, showed a relatively large range of elongation where the frequencies for small amplitudes are satisfied by an equation of the form

$$n = \frac{1}{2\pi}\sqrt{\frac{g[\alpha(1 + k\alpha) + 2\beta e]}{M}} \text{ (see Table VIII.),}$$

where, for a given wire α and β are constants, e is the elongation for the mass M, and e and M are connected by the relation $M = \alpha e + \beta e^2$. k is a small constant depending on temperature changes in the wire as the wire vibrates.

3. The modulus determined dynamically is, as usual, larger than that measured under static conditions (see Fig. 12).

4. This difference in moduli may be explained as the result of a large after-effect, the greater part of which disappears very rapidly after the removal of load (see Sec. V.).

5. The frequency decreases with increase of amplitude, in the case of copper and platinum-iridium. The relation may be represented by $n = n_0 - bA$ where b is a positive quantity. b is greater for large frequencies than for small (Tables V. and VI.). The variation for phosphor-bronze was exceedingly small, and for steel piano-wire n was evidently independent of the amplitude.

6. The wire which showed the greatest variation of period with amplitude also showed the greatest damping and the greatest after-effect.

7. The logarithmic decrement was measured and found to vary with the amplitude (Sec. VI.). For copper, at least, the increase of λ was very nearly proportional to increase of amplitude (Fig. 16). The logarithmic decrement does not vary with different frequencies in such a manner as to indicate that the damping is due to internal friction.

The writer is under obligation to Professor Guthe for proposing the investigation, and for making valuable suggestions during the progress of the work, to Professor Randall for advice in regard to some of the experimental part, and to Professor Curtiss for the use of a measuring engine of the Detroit Observatory.

Physical Laboratory,
University of Michigan,
March, 1914.

ROGER BACON AS A SCIENTIST.

BY KARL E. GUTHE.

IN the history of science of the thirteenth century two commanding figures stand out among all the natural philosophers of their time, Albrecht von Bollstaedt and Roger Bacon. Both have been highly praised as great men, and both have been sneered at as charlatans. Certainly neither of them had a high opinion of the other. Nothing appears to be more difficult than to decide upon the characteristics of a great scientist. In his book entitled *Grosse Männer* Ostwald selected only investigators who had made some remarkable discovery as examples of this type. But to measure all men by such a single standard as Ostwald's appears to me unjust and wholly onesided. I would rather call each of Ostwald's heroes a scientific genius and not restrict the class of great men so as to shut out any one who in one way or another has had a profound influence upon the progress of civilization.

Roger Bacon cannot be credited with a single epoch-making discovery, and yet he deserves to be called a great man. He is one of those rare scholars who combine with a remarkably extensive knowledge and with an admiration for the majesty and mystery of this world a powerful conviction that a certain unity underlies the various phenomena in nature and that all sciences—in their widest sense—are dependent upon one another. Such men attempt to rise above mere details, to view the world as a whole, and then present in bold outlines a picture of the Cosmos as they see it. In many cases their efforts result in the production of a mere cosmic encyclopedia, but for really great thinkers the world of observation blends with the world which they build up in their imagination to a unified picture, though possibly distorted and unreal when viewed by later generations.

We seem to be unable to get at any one time a complete view of this world; in fact our view is constantly changing, and every attempt to chain down the Cosmos has been unsuccessful. In this respect Roger Bacon achieved no more than others have before or after him. His works, like all similar attempts, should be entitled "Facts and Fancies." In view of the meager equipment of facts which the scientific world possessed at that time, fancies play an important rôle in Bacon's treatises. As a source of scientific information his writings are of little value, but as human documents describing the knowledge, beliefs and superstitions then existing, and showing the attitude of scholars towards scientific questions, they are of absorbing interest.

Man has not changed appreciably in 700 years. Much as we now discuss the value or uselessness of certain disciplines, so did men in Bacon's time. Just as at present we often consider any one an ignoramus who happens to disagree with us, and declare another a profound scholar whose views coincide with ours, so Bacon, who was a little more outspoken than is considered proper for a twentieth century professor, considered only one mental attitude towards important philosophic questions to be correct; and this indeed is the attitude which in the present scientific age is shared by many of us. Therefore we are liable to overestimate Bacon's importance in the history of science. He was not generally considered a great man in his own age and for some centuries afterwards, when the methods for searching after truth were different.

How was it possible that a man like Roger Bacon could arise in the thirteenth century? How could he have shaped for himself a world picture so different from that taught in the powerful University of Paris? I do not believe that even a master mind, such as he doubtless was, can create a great thought or state a fundamental truth without considerable preliminary work and study on the part of some predecessors who have at least felt it, though unable to express it as clearly. Indeed, Roger Bacon seems to me to be the product of a perfectly definite movement and—unfortunately for science—its last great exponent for the time being.

It was not until shortly before Bacon's time that Europe became acquainted with Arabian scholarship which itself was a product of the cultures of Greece and India. While the Arabs preserved and taught Greek philosophy, they were mainly interested in the purely scientific results of Greek scholarship and laid special emphasis upon the development of the mathematical sciences.

It seems that a revival of mathematics occurred first in England. Adelhard of Bath who had traveled among the Arabs for purposes of study, translated upon his return to England in 1130 the Elements of Euclid; Jean of Holywood wrote a book on arithmetic, and Robert Gross-tete, later Bishop of Lincoln, was greatly interested in mathematics and the sciences. So was Adam de Marisco. The last two, Roger's teachers of whose learning he speaks always with the greatest respect, must have been powerful and original minds and must have made a lasting impression upon the younger scholar.

Bacon was thus a product of a school of mathematics and formed his world picture accordingly. While he acknowledges the value of the languages—namely in order to avoid mistranslations—he says of mathematics: "He who knows not mathematics cannot know any other science; what is more, he cannot discover his own ignorance or find its proper remedies. So it is that the knowledge of this science prepares the mind and elevates it to a well authenticated knowledge of all things."[1]

This statement which seems highly appropriate as applied to physical sciences, refers, however, not only to the sciences proper, but to theology as well, as he explains at length in his *Opus tertium*: "Without mathematics we can not fix the dates of sacred history nor can we see the true relations between celestial and mundane occurrences."[2] Without geometry we cannot get a clear idea of the shape of Noah's ark, or of the tabernacle. Without arithmetic we cannot understand the symbolic meaning of the Trinity. Since at his time mathematics included also the science of music he adds as further argument, that without music we know nothing about hymns and invocations of spirits. A large portion of the *Opus tertium* is devoted to a discussion of the beneficial and elevating influence of music and rhythmical art. It may appear peculiar that Bacon places so much emphasis upon what might now be considered as trivial; but it should be remembered that these very things frequently formed the subject of highly learned disputations among Bacon's colleagues.

This man, so strongly convinced that mathematics forms the very foundation of all knowledge, went to *Paris* for further study; and what did he find? The far famed illustrious teachers of that time—Albertus Magnus, Thomas Aquinas and others—were interested in entirely different things. In Roger's estimation they could not compare with his former teachers who according to him were perfect in divine and human wisdom. The shortcomings of the Parisians he could easily explain by asserting: "The neglect of mathematics for thirty or forty years has nearly destroyed the entire study of Latin Christendom."[3]

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[1] *Opus majus*, pars IV ("Mathematicae in divinis utilitas," distinctio prima, ch. I): "Quoniam qui ignorat eam non potest scire caeteras scientias nec res hujus mundi, ut probabo. Et, quod pejus est, homines eam ignorantes non percipiunt suam ignorantiam, et ideo remedium non quaerunt. Ac per contrarium hujus scientiae notitia praeparat animum et elevat ad omnium certificatam cognitionem."

[2] *Opus tertium*, ch. XI: "Nam planum est quod sine mathematica non possunt sciri coelestia; et coelestia sunt causae rerum inferiorum, et causata non possunt sciri sine causis suis."

In disgust he turns away from his unsympathetic colleagues. He speaks in praise of only one person, and him indeed he calls a perfect mathematician. This man was a Picard by the name of Peter de Maharncuria. For many years historians have tried to identify him; Charles has shown him to be no other then Peter Peregrinus, the well-known writer of a letter on terrestrial magnetism and the compass.[4] This scholar was not a teacher in Paris, but an independent investigator, and apparently little known.

A man like Bacon would certainly be much interested in physical problems, and indeed he found great pleasure in the study of light which subject was at that time restricted to geometrical optics. In this field also he finds nothing in Paris. Twice there had been lectures in Oxford on light, but not once in Paris. Listen to his complaints about the conditions in the latter institution. "The man who pretends to be an authority in optics, knows nothing about its value, as clearly appears from his books; because neither has he written a treatise on this subject—and he would have done so, had he had the knowledge—nor has he said anything about it in other books." And then in his characteristic manner he closes with the words, "and therefor he cannot know any thing about philosophy."[5]

It is difficult to describe the peculiar charm which we find in the study of Bacon's works. We modern scientists are so accustomed—and I believe properly so—to eliminâte our own personality from our work, that our world has become one without feeling, that it speaks to us only through experimental facts. In Bacon's writings it is all so different. Those old scientists took no interest in the game when they eliminated sentiment and animosity from their discussions. At times it is almost uncanny to feel the old monk sitting right by your side and pleading with you in such a personal and direct manner. One can not help fearing that one's estimate of his proper place in the history of science might become warped by sympathy for him. It is difficult not to be persuaded that the great Albert was after all an impostor, wholly unfit to be a professor in the great university. And yet, Bacon's estimate of his great scholastic contemporaries was certainly too harsh.

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[3] *Opus majus,* pars IV, ch. I: "Et harum scientiarum porta et clavis est mathematica.—Cujus negligentia jam per triginta vel quadraginta annos destruxit totum studium Latinorum."

[4] A full account of this important work is given in Benjamin's *The Intellectual Rise in Electricity,* pp. 165-190.

[5] *Opus tertium,* ch. XI: "Haec autem scientia non est adhuc lecta Parisius, nec apud Latinos, nisi bis Oxoniae in Anglia; et non sunt tres qui sciant ejus potestatem: unde ille, qui fecit se auctorem, de quo superius dixi, nihil novit de hujus scientiae potestate, sicut apparet in libris suis, quia nec fecit librum de hac scientia, et fecissit si scivisset, nec in libris aliis aliquid de hac scientia recitavit.—Et ideo non potest scire aliquid de sapientia philosophiae."

We must not forget that science had just been revived in the Christian world. In 1209 the study of the books of Aristotle was forbidden; in 1215 Robert de Courçon, a papal legate, renewed the prohibition, expressly including Aristotle's metaphysics; in 1231 a bull of Gregory IX modified this decision with the proviso that the prohibited books were not to be used until they were proved to be free of error; and only 13 years before Bacon wrote his *Opus majus* was the ban raised. It is therefore evident that considerable time was needed for the assimilation of the new knowledge and for a slow growing appreciation of its method.

A thousand years before Bacon, Eusebius said with regard to scientific questions: "It is not through ignorance of the things admired by them, but through contempt of their own useless labor, that we think little of these matters, turning our souls to the exercise of better things." The attitude of the church had not changed in the least, for Bacon says: "When the philosophers are told in these days that they ought to study perspective, or geometry, or the languages, they ask with a sneer: What is the use of these tings? insinuating their uselessness. They refuse to hear a word said with reference to their utility; they neglect and condemn the sciences of which they are ignorant."[6]

We can easily understand that a man of Bacon's temperament and independence had no use for Albertus Magnus who prided himself that he taught the ancient science in such a way that no one could recognize his own personal views. The animosity shown in the above quotation is a much more serious matter than a disagreement of two scholars would be at the present time. Not two men, but two schools of thought were battling with each other. Each had devised a method for the assimilation of the new knowledge, and either school used all its influence to suppress the other. Personally I am convinced that Roger Bacon would not have hesitated a moment to silence the great Albert and his adherents, had he had the power to do so. At that time a compromise was impossible; freedom of thought had not yet been discovered.

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[6] *Opus tertium*, ch. VI: "Nam philosophantes his diebus, quando dicitur eis quod sciant perspectivam, aut geometriam, aut linguas, et alia multa, quaerunt cum derisione, 'Quid valent haec?' asserentes quod inutilia sunt. Nec volunt audire sermonemde utilitate; et ideo neglegunt et contemnunt scientias quas ignorant."

If now we consider Bacon's actual accomplishments in science a little more closely we find nothing very remarkable, though we cannot help but admire his encyclopedic knowledge of details and the clear grasp of their interrelation.

In mathematics he has added nothing to knowledge, as far as we can see. No mathematical treatise which might form a part of his contemplated all-embracing work has ever been found. This is disappointing, for Roger states that the quadrature of the circle for which the ancients had searched in vain, had finally been accomplished.[7] It would have been interesting to compare his proof of this fallacy with that of Cusanus, 150 years later.

Bacon used his knowledge of astronomy to make an immense number of clever calculations, especially with a view of fixing the dates of occurrences mentioned in the Bible. Thus he proves to his perfect satisfaction that creation took place in the fall of the year,[8] and he knows the exact date when the flood began. He also calculated the size of the earth. This was a rather complicated problem. The decimal system was not in general use at that time and all the data which he had at his disposal were that the length of one degree on the earth's surface was fifty-six miles plus two-thirds of a mile plus twenty-seven ninetieths plus one six hundred and thirtieth of a mile. It needs real mathematical gymnastics to reach a solution. His final result was not very far from the truth; in fact, it was nearly as accurate as that on which 400 years later (in 1666) Newton according to tradition attempted to verify his law of gravitation with the result that he did not dare publish this fundamental law, until Picard 18 years later had made new and more accurate measurements.[9] Bacon found further that the diameter of the earth was $3\frac{2}{5}$ times the diameter of the moon, a value not bad as an approximation; but when he got farther away from mother earth he was less successful, for he found that the diameter of the sun was only 5½ times greater than that of the earth.

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[7] *Opus majus, pars* I, ch. VI: "Nam quadraturam circuli se ignorasse confitetur, quod his diebus scitur veraciter."

[8] *Opus majus,* pars IV, (*loc. cit.*): "Nam multi voluerunt secundum sententiam vulgi, quod mundus fuerit creatus circiter aequinoctium vernale; sed alii apud aequinoctium autumnale; quia in veritate secundum Hebraicum veritatem, annus, quantum ad seriem temporis naturalem, incipit circiter aequinoctium autumnale."

[9] This is merely a tradition. Newton's difficulty seems to have been of an entirely different nature. See Cajori, *History of Physics,* p. 58.

In spite of some mistakes of this sort, he was correct in many other calculations. He very earnestly and impressively urged a change of the calendar by showing clearly that Christian chronology did not agree with astronomical observations. Though he calls this error in itself "*horribilis*," this is nothing in comparison with the fact that the whole order of ecclesiastic festivals is confounded. It seems to him inexcusable that in 1267 Lent began and ended a week too late and that therefore the infidels, Arabs, Hebrews and Greeks pointed with abhorrence at the stupidity of the Christians.[10]

Bacon was by no means the first to advocate an improvement of the calendar, though this is sometimes claimed by his admirers. He himself mentions Theophilus, Eusebius, Victorinus, Cyrillus, Beda and others who had before him labored for the same reform. Neither was Bacon successful. The necessary change was not made until more than 300 years later.

We should not blame Bacon for believing implicitly in astrology. All the best thinkers of his time did so and even his great namesake, three centuries and a half after Roger, had not abandoned it although the Copernican theory had at that time been accepted by the leading astronomers.

As we pass on to physics we find that in this science also Bacon's contributions of new facts are very meager. At his time physics consisted of the most elementary mechanics and what we call now geometrical optics. Very little was known of electricity and magnetism. While he was an admirer of Aristotle and claims to have read all his books, he was not a blind follower like Albertus Magnus. In fact he praises Robert of Lincoln for disregarding the writings of Aristotle and working out his theories independently. Besides large portions of the *Opus majus* ("*De scientia perspectiva*") and *Opus tertium* devoted to optics his main work on this subject is "*De multiplicatione specierum.*"

Roger Bacon knew the law of reflection and also that light may be refracted, though of course he was unacquainted with the law of refraction which was not discovered until 1621, by Willibrod Snell. He was much interested in the practical application of reflection by means of spherical mirrors, but did not know how to find their focus. Nevertheless he believed that by experiments in this direction much might be achieved. His friend Peter Peregrinus had already worked on such mirrors for three years and Bacon hoped that he would soon be able to perfect burning mirrors of great power. Of what enormous value would these be, he exclaimed. The armies of the Saracens and other enemies of Christendom could be burned at long distances by a dozen of such mirrors, attended to by a scientist and two helpers, and thus much bloodshed would be prevented.[11] What an elegant and inexpensive method of abolishing war and establishing universal peace!

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[10] *Opus majus*, pars IV (*loc. cit.*): "Nam omnes literati in computo et astronomi sciunt haec, et derident ignorantiam praelatorum qui haec sustinent. Atque philosophi infideles, Arabes, Hebraei, et Graeci, qui habitant inter Christianos ut in Hispania et Aegypto et in partibus orientis, et in multis aliis mundi regionibus abhorrent stultitiam quam conspiciunt in ordinatione temporum quibus utuntur Christiani in suis solemnitatibus."

Bacon's knowledge has been much overestimated. Some authors thought he invented the telescope. This is impossible. He did not even know that the greatest magnifying effect of a simple lens is obtained by holding it near the eye; for like Alhazen 200 years before him, he advised that the lens be laid upon the object to be viewed through the glass.[12] The great exponent of experimental work must have copied this mechanically without making independent observations. He knew, however, that the magnifying power depends upon the angle under which an object appears and he dreamed of the time when by a combination of lenses this angle might be increased so much that we might read the smallest script at incredible distance, that a boy would appear as a giant, that a small body of soldiers would be seen as a large army. By making the images of sun, moon and stars descend upon the heads of enemies, they would flee terror-stricken.[13] To distort this sentence as meaning that Bacon actually knew a telescope or microscope, is rather daring. He merely stated a problem which he hoped might be solved by future generations.

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[11] *Opus tertium,* ch. XXXVI: "Certe si duodecim talia specula haberent Aconenses, et illi qui sunt ultra mare Christiani, ipse sine effusione sanguinis pellerent Saracenos de finibus eorum: nec oportet Dominum regem Franciae cum exercitu transire pro illa terra acquirenda. Et quando ibit, plus valeret ei habere illum magistrum cum duobus aliis, quam majorem partem exercitus sui, ne dicam totum exercitum."

[12] *Opus majus,* pars V, ("De scientia perspectiva," pars III, 2, ch. IV): "Si vero homo aspiciat literas et alias res minutas per medium crystalli vel vitri vel alterius perspicui suppositi literis, et sit portio minor spherae cujus convexitas sit versus oculum, et oculus sit in aere, longe melius videbit literas et apparebunt ei majores."

[13] *Ibid.*, ultima distinctio, ch. IV: "Nam possumus sic figurare perspicua, et taliter ea ordinare respectu nostri visus et rerum, quod frangentur radii et flectentur quorsumcunque voluerimus, ut sub quocunque angulo voluerimus videbimus rem prope vel longe. Et sic ex incredibili distantia legeremus literas minutissimas et pulveres ac arenas numeraremus propter magnitudinem anguli sub quo videremus, et maxima corpora de prope vix videremus propter parvitatem anguli sub quo videremus, nam distantia non facit ad hujusmodi visiones nisi per accidens, sed quantitas anguli. Et sic posset puer apparere gigas, et unus homo videri mons, et in quacunque quantitate—sic etiam faceremus solem et lunam et stellas descendere secundum apparentiam hic inferius, et similiter super capita inimicorum apparere et multa consimilia, ut animus mortalis ignorans veritatem non posset sustinere."

It is this almost inspired presentiment of the coming development of science, this scientific instinct, which makes Bacon so interesting to later generations. While reading his optical works I found a most unsuspected treasure, namely what I believe to be the first appearance of the undulatory theory of light, in a rather crude form, it is true, but easily recognized. It is his theory of the propagation of species. The translators speak of Bacon's theory of propagation of "force," a word whose exact meaning as a physical quantity bothers us physicists even at the present time. But his species is not a force at all, but a quantity, as flexible and unreal as *our* much admired electromagnetic vibration of the ether. It is something caused by the acting body[14]—in the case of light, by the luminous body, from which the species proceeds into space in straight lines. While there may be a resemblance to the forms or replicas which according to Lucretius and the earlier philosophers proceeded from all luminous bodies, it would be an entire misapprehension of Bacon's views to identify these forms with his species. The species is not a part of the acting body, i. e., the body which we see; but it is generated in the surrounding medium. The luminous source acts on the first portion of the medium, stimulating its latent energy to the generation of the species. This portion, thus transmuted, acts on the part of the medium next succeeding, and so the action proceeds from point to point.[15] Species has therefore no bodily existence apart from that of the medium through which it passes, and light cannot be a material body.[16] Moreover each species lasts only an imperceptible time, since the medium has a nature opposed to the creation of species and destroys it, but not until it has affected in a similar manner the surrounding medium. The luminous body must therefore cause one species after another in very rapid succession, but since they take place in time, it should be possible to count them.[17] Finally the propagation of species through space requires time, for otherwise light could be at once at the beginning of space, at its middle and at the end at the same time; which is a property of the Creator alone and not of any created thing.[18]

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[14] *Multiplicatio specierum*, pars I, ch. I: "Species autem non sumitur hic pro quanto universali apud Porphyrium, sed transumitur hoc nomen ad designandum primum effectum cujuslibet naturaliter."

[15] *Ibid.*, pars I, ch. III: "Quod non potest species exire nec emitti ab ipso agente, quia accidens non permutat subjectum, nec pars substantialis sine corruptione substantiae totius.—Sed species est effectus agentis naturalis, et naturaliter productus est, quare species ipsa debet de potentia materiae generari.—Unde forma ignis non alterat materiam vel alterius, ergo non potest species facta in prima parte patientis alterare illam partem ad alium effectum producendum in ea, sed partem secundam. Et ita quae fit in secunda alterabit tertiam, et sic ulterius."

[16] *Ibid.*, pars III, ch. I: "In primo consideratur an species sit corpus veraciter, sicut multi posuerunt. Quod vero non sit corpus probatur per hoc, quod non dividit latera continentis medii, quod est locum in alio occupare, ut omnes sciunt. Et ideo si species esset corpus secundum se, essent duo corpora simul, quod non est possibile."

How much like the more modern undulatory theory of light this is. The species are periodic disturbances impressed upon the medium which due to its "contrary disposition" opposes them, but at the same time hands on the disturbance from point to point with a great, though finite, velocity. The carrier of the disturbances is an ether endowed with such physical properties as suited Bacon's purpose and as real as any of its many successors. The existence of a vacuum was to him an impossibility.[19] To find such a theory, even in a crude form in a work written six and one-half centuries ago, seems remarkable indeed. And still more, In his search after a unified view, Bacon extends the meaning of species to include the action of gravitational and other forces. It almost appears as if he had felt instinctively that there must be such a thing as that which we now call energy.

Personally I do not believe that Bacon was a sufficiently deep thinker to have originated the above theory, mixed up, as it is, with crude generalizations which are characteristically Baconian. His works show clearly a remarkable power of adaptation to Arabian scholarship, and I hope that further study may allow me to trace the "undulatory theory" back to an earlier author.

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[17] *Ibid.,* pars VI, ch. I: "Dicto de generatione speciei et multiplicatione, nunc dicendum est de corruptione. Et patet eam esse corruptibilem, quia est generabilis.....Caeterum natura patientis specifica nata est ad contrariam speciei, si contarium habeat, vel ad dispositionem contrariam illi quae per speciem inducitur....et sic per consequens species lucis vel alterius corrumpitur per accidens per contrarium, etsi non per se."

Ibid., ch. III: "Deinde tertium consideratum est quod cum idem agens iterum redeat super eandem naturam patientis, facit impressionem seu speciem diversam numero a priore, et ideo effectus numeratur."

[18] *Opus tertium,* ch. XLII: "Iterum, nulla virtus agit in instanti; sicut probatur sexto Physicorum.....Iterum, nihil, potest simul et semel esse in diversis locis, nisi Creator. Sed si in instanti fieret tunc esset simul et semel in principio spatii, et in medio, et in fine, et per consequens in omnibus partibus spatii; ergo non esset creatura."

[19] *Opus majus,* pars V ("De scientia perspectiva," pars I, distinctio nona): "Sed si vacuum poneremus inter coelum et terram, nec esset densum nec rarum.....Atque vacuum non habet aliquan naturam, unde impediat speciem, nec unde resistat speciei, quia nulla natura est ibi....species enim est res naturalis et ideo indiget medio naturali, sed in vacuo nulla natura est."

Bacon was a man of great imagination, as every scientist ought to be. While our modern scientific dreams are checked on all sides by innumerable laws of nature which they must *obey*, *he* could give his fancy free rein. Listen to some of his prophecies:

"Ships will be built which, with a single man steering them, will move through rivers and the ocean with a greater speed than if they were filled with oarsmen. Carriages can be made to move with incredible speed without the help of any animals. Flying machines will be constructed so that one man sits in the middle of the apparatus, revolving some ingenious device by means of which wings beat the air after the manner of flying birds."[20]

In spite of all these speculations Bacon does not lay great emphasis upon them. In fact they are not found in his more serious work. Though he had quite a reputation as an alchemist and a magician, he holds all magic in contempt. In describing some astonishing experiment with the magnet, he says with fine humor: "Magicians make this experiment, mumbling incantations and believing that things happen by virtue of their songs. I have neglected chanting and have understood the marvelous work of nature."[21] He clearly recognized that a science, based upon superstition, speculation and arm-chair philosophy, cannot be of permanent value. But just this kind of science was taught at his time, disputations were held about the meaning of infinity, or in what language angels converse with each other, or how many angels can stand upon the point of a needle.

Bacon's fame does not rest upon any discovery he may have made, nor upon his actual knowledge of scientific facts, nor upon his more or less correct interpretation of human experiences, but upon the fact that more than any other of the early scholars he emphasized that none of the sciences could make any progress without the application of what he terms the "*scientia particularis*," namely experimental science. "All sciences," he says, "are connected; they lend each other material aid as parts of one great whole, each doing its own work, not for itself alone, but for the other parts; as the eye guides the body and the foot sustains it and leads it from place to place."[22] "But above all these there is one, more perfect than any, which they all serve, namely experimental science. It alone can test their conclusions, which cannot be done by mere argument."[23]

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[20] *De secretis operibus artis et naturae, et de nullitate magiae,* ch. IV: "Nam instrumenta navigandi possunt fieri sine hominibus remigantibus, ut naves maximae, fluviales et marinae, ferantur unico homine regente, majori velocitate quam si plenae essent hominibus. Item currus possunt fieri ut sine animali moveantur cum impetu inaestimabili, ut aestimamus currus falcati fuisse, quibus antiquitus pugnabatur. Item possunt fieri instrumenta volandi, ut homo sedeat in medio instrumenti revolvens aliquod ingenium, per quod alae artificialiter compositae aerem verberent, ad modum avis volantis. Item instrumentum parvum in quantitate ad elevandum et deprimendum pondera quasi infinita, quod nihil utilius est in casu."

[21] *Opus majus*, pars VI ("Scientia experimentalis," ch. XII): "Et ideo magici utuntur hoc experimento, et dicunt carmina diversa, et credunt quod ex virtute carminum istud contingat. Et ego neglexi carmina et inveni opus naturae mirabile."

"Experimental science has three great prerogatives among other sciences: First, she tests by experiment their noblest conclusions; next, she, the sole mistress of speculative sciences, discovers magnificent truth to which these sciences of themselves can by no means attain; her third dignity is, that she by her own power and without respect to other sciences investigates the secrets of nature."[24]

The examples which Bacon gives for these prerogatives are very curious and amusing. The first is a—for that time admirable—research as to the nature of the rainbow, though the moment he leaves the solid ground of experimentation he falls into error. For example, he says there can be only five colors in the rainbow, because five is a more perfect number than seven, the number which Aristotle had chosen.[25] While he experiments skilfully with reflection and refraction of light he reaches amusing conclusions: The direct ray is the most perfect, pertaining to the nature of God; then come the refracted rays corresponding to the vision of angels; while we poor mortals must be content with the weakest of them all, namely with a vision by reflected rays. For the apostle Paul says: Now we see through a mirror darkly, but then from face to face.[26]

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[22] *Opus tertium*, ch. IV: "Nam omnes scientiae sunt connexae, et mutuis se fovent auxiliis, sicut partes ejusdem totius, quarum quaelibet opus suum peragit, non solum propter se, sed pro aliis: ut oculus totum corpus dirigit, et pes totum sustentat, et de loco ad locum deducit; et sic de aliis."

[23] *Ibid.*, ch. XIII: "Sed praeter has scientias est una perfectior omnibus, cui omnes famulantur, et quae omnes miro modo certificat; et haec vocatur scientia experimentalis, quae negligit argumenta, quoniam non certificant, quantumcunque sint fortia, nisi simul adsit experientia conclusionis, ut ostendo in tractatu de ista scientia. Et ideo haec docet experiri conclusiones nobiles omnium scientiarum, quae in aliis scientiis aut probantur per argumenta, aut investigantur per experientias naturales et imperfectas."

[24] See Whewell's *History of the Inductive Sciences,* Vol. I, p. 375.

[25] *Opus majus,* pars VI, ch. XII: "Quum enim Aristoteles dicit in Sensu et Sensato septem esse colores....sed quinque principales colores sunt per naturam distincti. Nam quinarius est melior numeris omnibus, ut Aristoteles dicit in libro Secretorum....Et quia numerus quinarius res certius distinguit et melius, ut dictum est, ideo natura magis intendit quinque colores."

[26] *Opus majus,* pars V: ("De scientia perspectiva," pars III, ultima distinctio, ch. II): "Aliter vero triplicatur visio secundum quod fit recte, fracte,

The examples of the second prerogative of experimental science are three: the art of making an artificial sphere which shall move with the heavens by natural influences, i. e., a *perpetuum mobile*. This was the great invention described by Bacon's friend, Peter Peregrinus. Secondly, the art of prolonging life, which experiment may teach, though medicine has no means of securing it, except by regimen. Thirdly, the art of making gold, finer than fine gold, which goes beyond the power of alchemy.

The third prerogative of experimental science, arts independent of received sciences, is shown by curious examples, many of them whimsical traditions. Thus it is said that the character of a people may be changed by altering the air. This refers to the answer which Aristotle is said to have given Alexander who wanted to know what he should do with certain barbarous nations. The reply was: If you can alter the air, permit them to live; if not, put them to death.

Arguments like these, should not, however, prejudice us against Bacon and the real service which he has rendered science. He outlines a definite method; he points to the only way in which progress may be achieved; and it is this service which entitles him to an honored place in history. It is true that there were experimental scientists before him, Ptolemy, Alhazen and many others, but none of them has spoken so clearly of the supreme importance of experiment, as he whose fanciful speculations appear childish in the better knowledge of to-day. In his appreciation of experimental demonstrations Roger Bacon was 300 years ahead of his time; he anticipated the scientific renaissance of the sixteenth century. Indeed, I place him in this respect far above the second Bacon though the latter managed to earn greater fame. But Roger Bacon was after all a child of his time. He was an astrologer and an alchemist, and his arguments did not differ to any marked extent from those employed by the despised teachers of the university of Paris. His own knowledge and accomplishments were advertised by him as unblushingly as by other learned men of his time. He could teach in three or six months all that he himself had learned in forty years of con-

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et reflexe. Prima est perfectior aliis, et secunda certior est, tertia incertissima. ....nam rectitudo visionis Deo debetur; declinatio a rectitudine per fractionem, quae debilior est, angelicae naturae convenit: reflexiva visio, quae est debilior, homini potest assignari.....Et homo habet triplicem visionem, unam perfectam, quae erit in statu gloriae post resurrectionem; aliam in anima separata a corpore in coelo usque ad resurrectionem, quae debilior est; tertiam in hac vita, quae debilissima est, et haec est recte per reflexionem. Secundum quod dicit apostolus 'videmus nunc per speculum in aenigmate, sed in gloria a facie ad faciem.'"

tinuous study; he would even undertake to teach Hebrew or Greek in three days.[27] Though his name was not mentioned in learned treatises of later times he must have had a number of secret admirers; for what better proof could we ask for than the fact that his writings have frequently been literally copied without any credit being given to him. Scientific plagiarism does not seem to have been a crime in those times, and I believe that *Francis* Bacon practised it cheerfully and extensively after some good fortune had made him acquainted with the works of the old monk, his namesake.

Not until the last century has Roger Bacon been shown to be the real author of much wisdom attributed formerly to others. Let me close with an interesting example. All that Christopher Columbus knew of Greek and Roman authors; all references of Aristotle, Strabo, and Seneca as to the proximity of Eastern Asia to the pillars of Hercules, references, which according to Columbus's son, Don Fernando, induced his father to look for the discovery of the East Indies—all this the admiral learned from the writings of cardinal Alliacus (Peter d'Ailly). He carried them with him on his travels; he translated in a letter from Haiti, addressed to the Spanish monarch a part of Alliacus's treatise *De quantitate terrae habitabilis.* Little did he know that Alliacus in his turn, had copied this, almost word for word, from the *Opus majus* of Roger Bacon.[28]

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[27] *Opus tertium,* ch. XX: "Multum laboravi in scientiis et linguis, ut posui jam quadraginta annos postquam dedici primo alphabetum; et fui semper studiosus....et tamen certus sum quod infra quartam anni, aut dimidium anni, ego docerem ore meo hominem sollicitum et confidentem, quicquid scio de potestate scientiarum et linguarum....sed certum est mihi quod infra tres dies ego quemcunque diligentem et confidentem docerem Hebraeum, ut sciret legere et intelligere quicquid sancti dicunt....Et per tres dies sciret de Graeco iterum; et non solum sciret legere et intelligere quicquid pertinet ad theologiam, sed ad philosophiam et ad linguam Latinam."

[28] Humboldt's *Kosmos,* Vol. II.